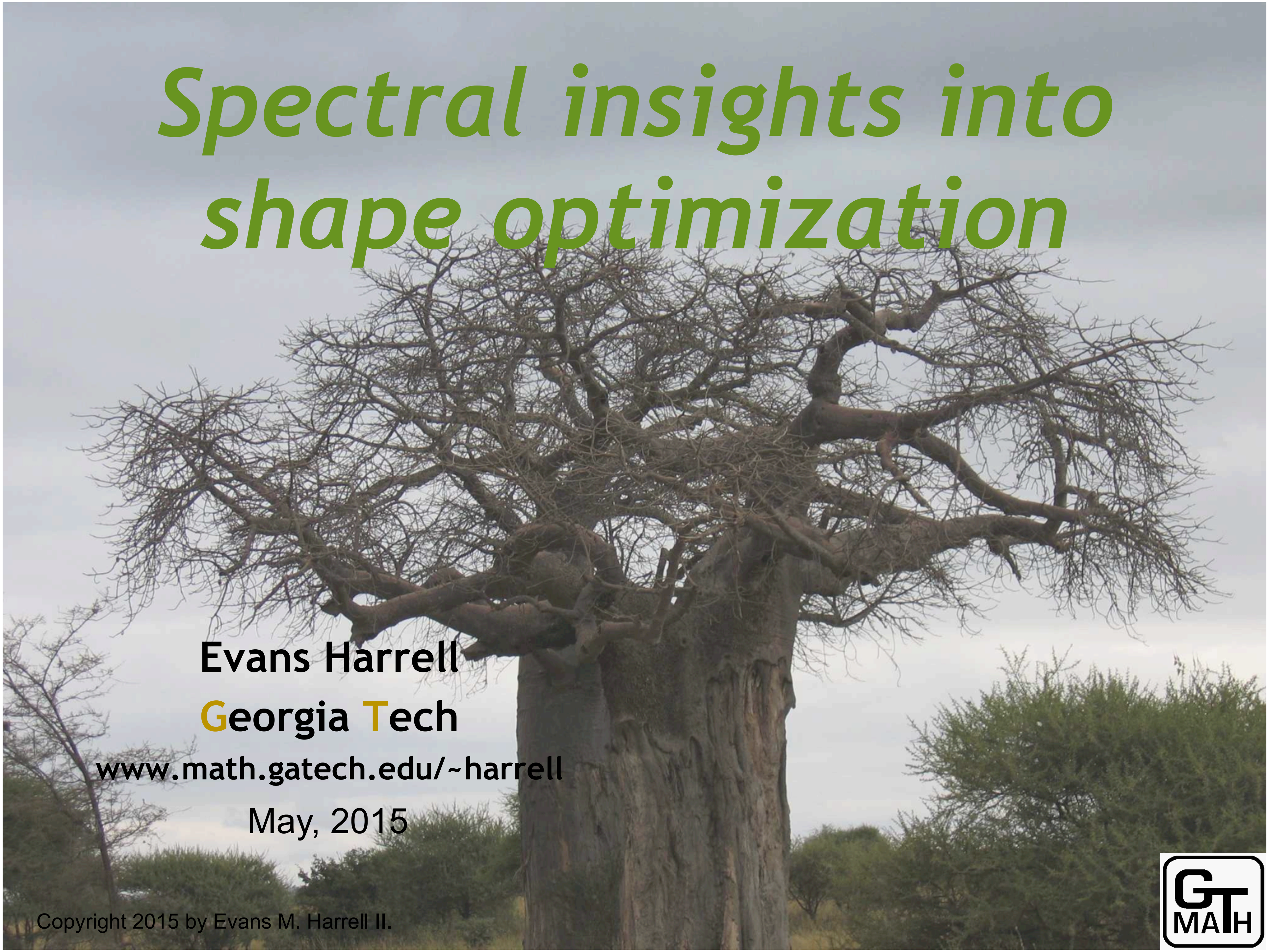

Spectral insights into shape optimization

Evans Harrell

Georgia Tech

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May, 2015

Why has Georgia Tech come to Africa?

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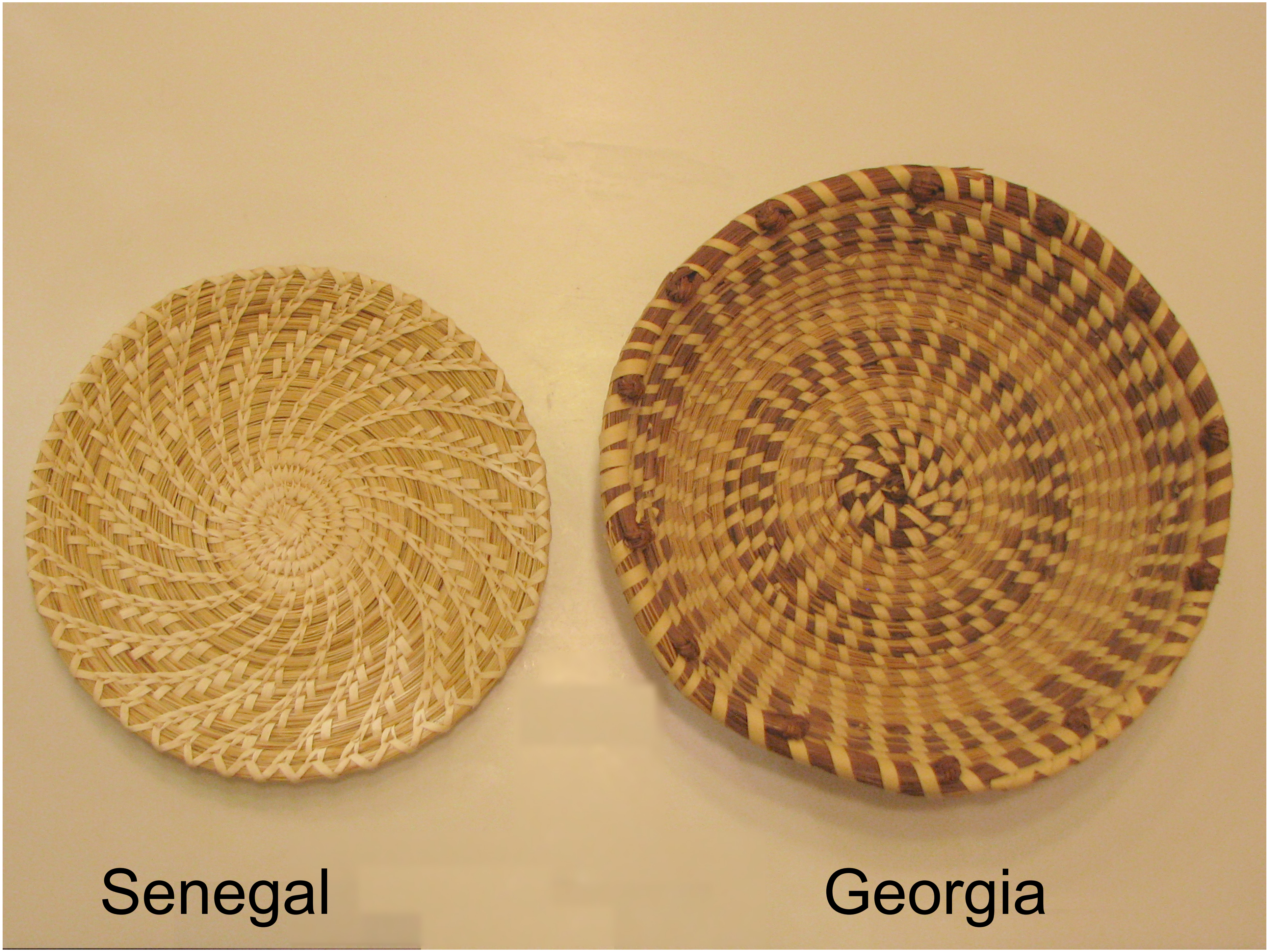
- The State of Georgia and Georgia Tech have had close connections to Africa for a long time.

AFRICA ATLANTA

2014

http://africaatlanta.org/


Senegal


Senegal
Georgia

GT AFRICA PROJECTS, 2014

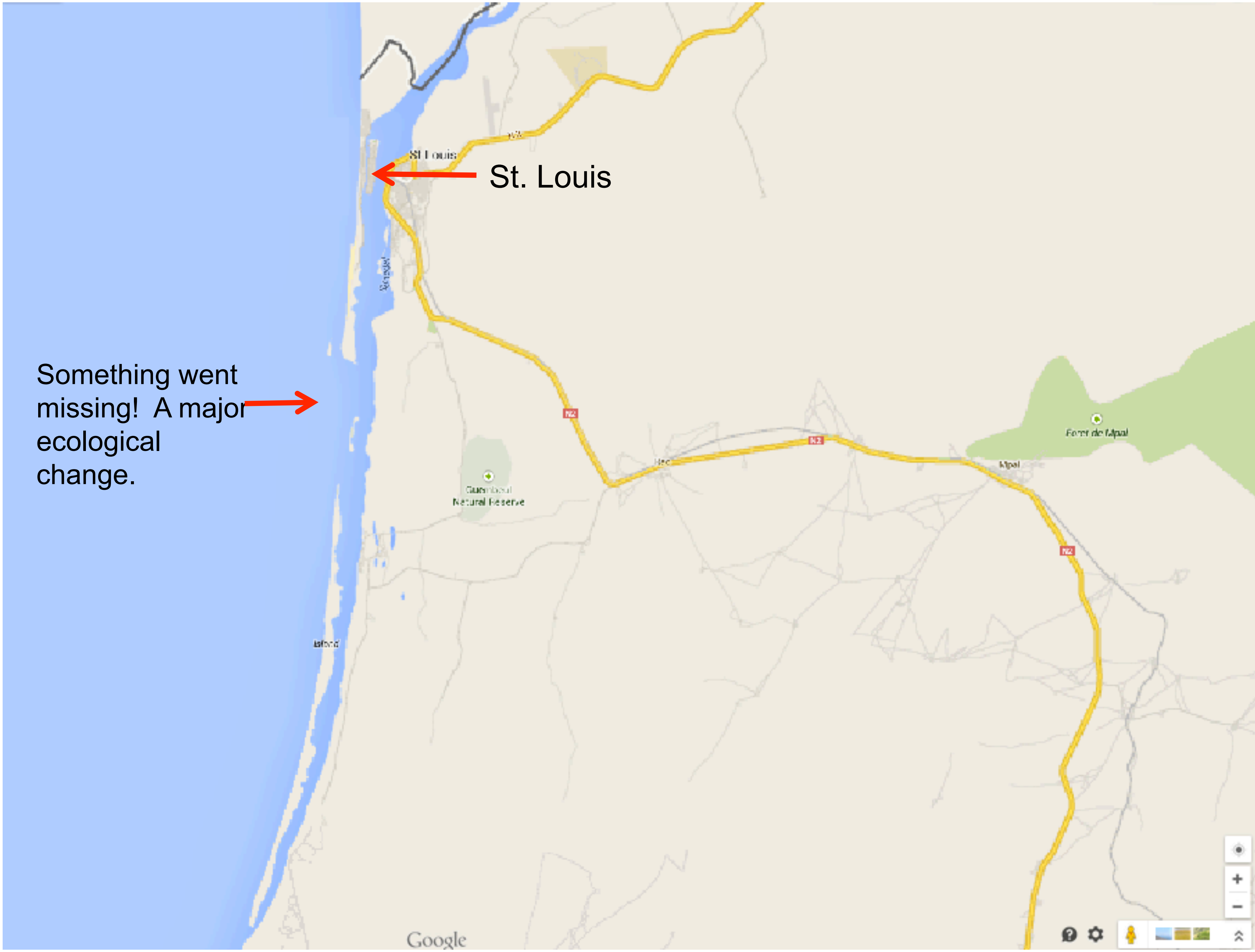
Faculty/Resera	Project Summ	GT School	GT College	Country of Project	Faculty/Reseracher's Name				
Ahamad, Must	Information M	CS	COC	Sudan	Hughes, Josep	Water assessm	CE	COE	Angola
Ahamad, Must	Information se	CS	COC	?	Hunter, Mike	Monitoring Ele	CS	CoC	Sudan
Amekudzi, Adj	Sustainable De	CE	COE	Ghana	Ippolito, Christ	Representatior	ML	IAC	Francophone Africa
Akyildiz, Ian F.	Telecom	ECE	COE	South Africa	Kakeu, Johnson		ECON	IAC	Cameroon
Ammons, Jane	Uganda Milline	ISYE	COE	Uganda	Keskinocak, Pi	Humanitarin L	ISYE	COE	
Bartholdi, John	Supply chains	ISYE	COE	South Africa	Leon, Roberto	Earthquke Eng	CEE	COE	
Bauchspies, W	Sustainability,	HTS	IAC	West Africa	Lurie, Nicholas	Marketing	Mgt.	COM	Cameroon
Best, Michael	EHELD Bid	CC	COC	Liberia	McIntyre, John	CIBER Center	Mgt.	CoM	??
Caravati, Kevin	Cameroon water Assessment			Cameroon	McKnight, Phil	LBAT Program	Mod. LAN.	IAC	Egypt
Caravati, Kevin	Emory Center for Global Safe Water			Rwanda	Okereke, Raph	ME	COE	COE	Nigeria
Cherkaoui, Mo	Materials mod	MSE	COE	Morocco	Oyelere, Adegboyega, K.		Chemistry	COS	
Chu, Meh chu	Former Preside	ECE	COE	Cameroon	Oyelere, Ruth	School and HI	ECON	IAC	Nigeria
Cozzens, Susa	Nanotechnolog	Public Policy	IAC		Schatz, Michae	Hands-on Rese	Physics	COS	Cameroon
Cozzens, Susa	Doctoral Progr	Public Policy	IAC		Streelman, Jef	Lake Malawai	Biology	COS	Malawi
Ergum, Ozlem	Humanitarin L	ISYE	COE		Swann, Julie	Humanitarin L	ISYE	COE	
Esogbue, Augu	Advisor, GT Sc	ISYE	COE		Ragauskas, Ar	Utilization of b	Chemistry	COS	Kenya
Farooq, Nihad,	Diaspora Studi	LCC	IAC		Taillefert, Mart	Oceanography	EAS	COS	
Fernandez, Fac	Drug quality st	Chemistry	COS	various locations	Thomas, Valer	Renewable ene	ISYE	COE	South Africa
Foote, Andrew	Cameroon Water Project				Thomas, Valer	Air pollution cc	ISYE	COE	South Africa
Gangbo, Wilfre	Applied Mathe	Math	COS	Benin, Senegal, M	Uwaifo, Ruth C	Sustainability,	ECON	IAC	West/Central Africa
Georgakakos,	Graduate Wate	CEE	COE	South Africa	Walker, Bruce	STEM Educatic	Psyc.	COS	Kenya
Georgakakos,	Nile Decision S	CEE	COE	Burundi, Congo, E	.				
Georgakakos,	Environmental	CEE	COE	Democratic Repub	.				
Graham, Stuar	Community En	Mgt.	COM	Tanzania	.				
Gleason, Rudo	HIV & Cardiova	ME	COE	Ethiopia, Cameroo	.				
Goodman, Sey	Difusion & Abs	INTA	IAC	Ghana, Togo	.				
Goodman, Sey	Information se	INTA/GTISC	IAC/COC	Tunisia, Rwanda	.				
Goodman, Sey	Cybercrime, Ir	INT, GTISC	IAC, COC	Cape Verde	.				
Goodman, Sey	Establish PhD	INTA	IAC	Ethiopia	.				
Goodman, Sey	Cyber Crime a	IAC	CoC	Mauritius	.				
Harrell, Evans	Applied Mathe	Math	COS	Senegal, Tunisia	.				
Harrell, Fox	The Living Libe	LCC	IAC	Liberia	.				
Hughes, Josep	Water assessm	CE	COE	Angola	.				

(List is known to contain inaccuracies.)

Why has Georgia Tech come to Africa?

- The State of Georgia and Georgia Tech have had close connections to Africa for a long time.
- Where there are needs there are opportunities for technology, in particular mathematics.

Langue de Barbarie, 2003


St. Louis
Something went missing! A major ecological change.
St Louis
N2
Mpal
Google

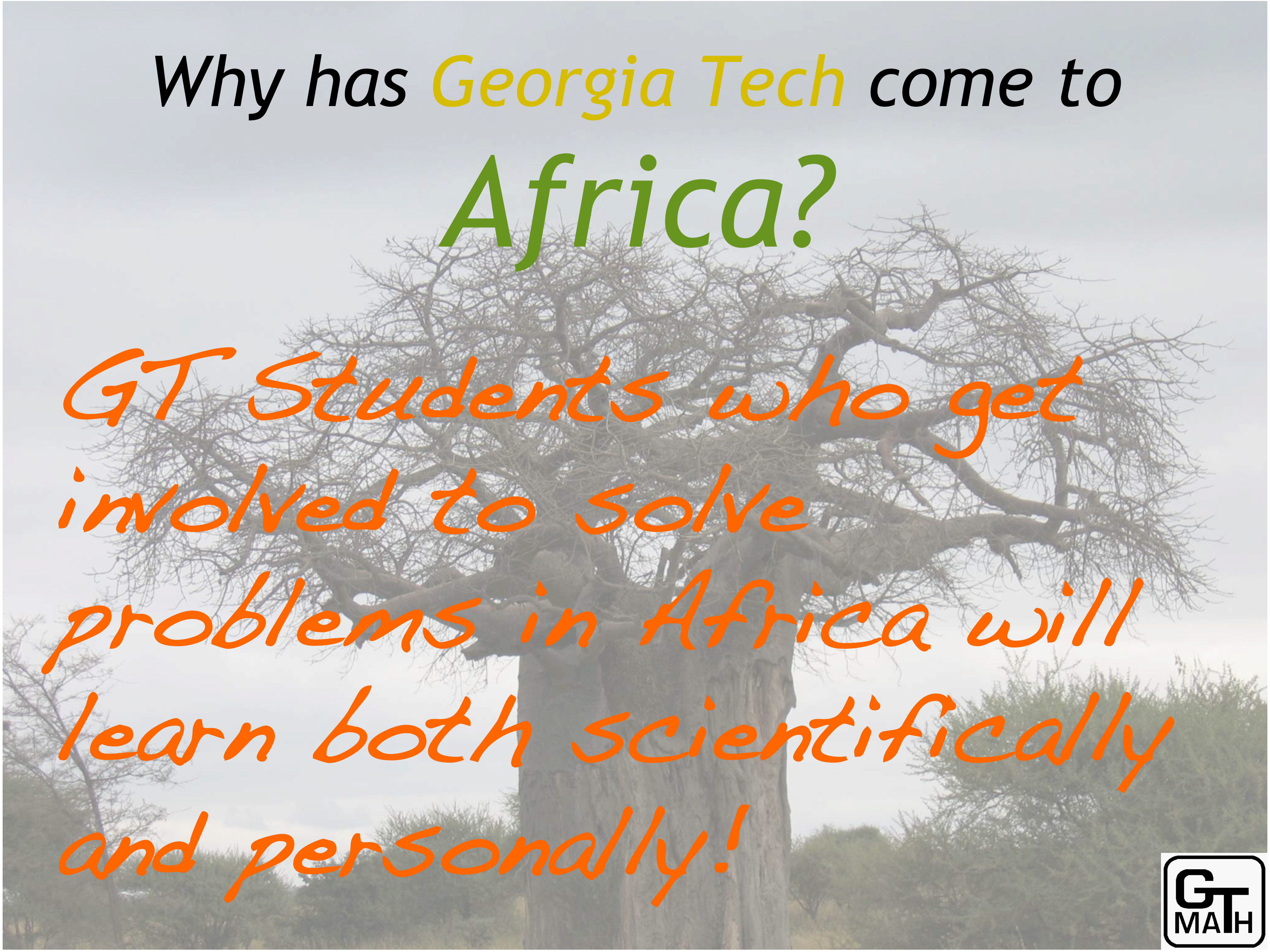
Langue de Barbarie : la brèche de l'espérance ?

Mary Teuw Niane, Université Gaston Berger, Sénégal, niane@ugb.sn
Abdou Sène, Université Gaston Berger, Sénégal, asene@ugb.sn
Saint-Louis, le 01 janvier 2004

L'ouverture d'une brèche sur la Langue de Barbarie dans la nuit du vendredi 03 au samedi 04 octobre 2003 a sauvé une bonne partie de la ville de Saint-Louis et ses environs d'une inondation catastrophique certaine. D'ailleurs, depuis la mi-septembre certaines localités de Gandiole comme Pilot étaient sous les eaux et certaines zones de Pikine, Diameguène, Léona et Darou gardent encore les stigmates d'une saison des pluies certes tardive mais ayant donné de fortes averses et par conséquent des flaques très persistantes.

A la joie légitime et débordante des populations et des autorités de voir le niveau du fleuve baisser inexorablement a succédé l'étonnement voire l'inquiétude de constater qu'en plein mois de novembre les eaux du fleuve Sénégal s'étaient retirées de plusieurs arches et des enfants jouaient en toute innocence et gaieté sur une bonne partie du lit du fleuve. Ce spectacle presque


Langue de Barbarie, 2005


Why has Georgia Tech come to
Africa?
GT Students who get involved to solve problems in Africa will learn both scientifically and personally!
GT MATH

Why has Georgia Tech come to Africa?

- The State of Georgia and Georgia Tech have had close connections to Africa for a long time.
- Where there are needs there are opportunities for technology, in particular mathematics.
- Opportunities for science and education will greatly increase in Africa.

The Next Einstein Project

www.aims-senegal.sn

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- AIMS-Journal-Club
- Next Einstein Initiative
- Essay Proposal
- Events
- Recruitment
- worth knowing

News

Workshop on "'Mathematical tools for understanding and managing fisheries : Synthesizing and refining data and models'" du 07 au 09 Mai 2014, see more details!!!!

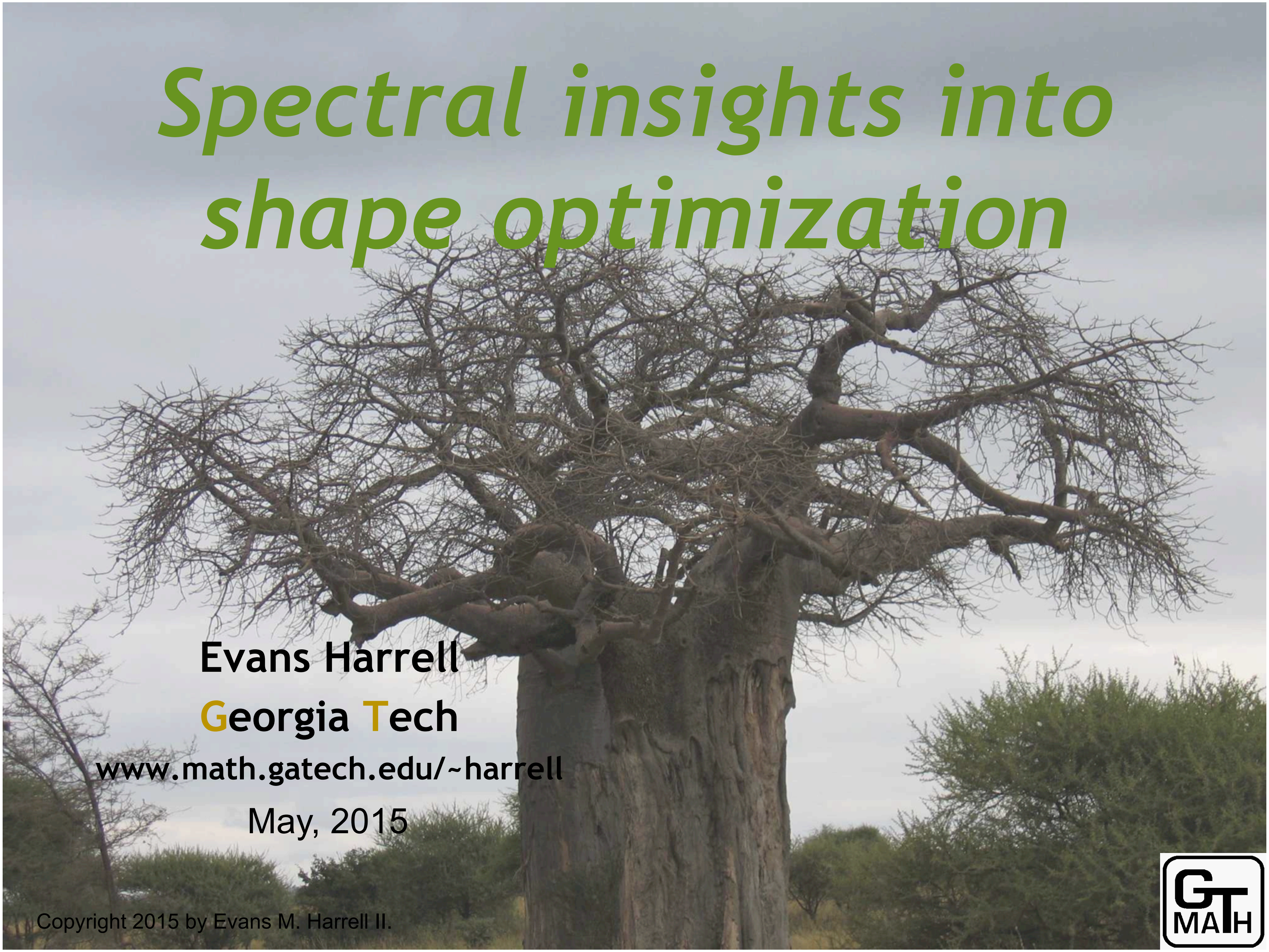
Mr. Mary Teuw Niane (Minister for Higher Education and Research in Senegal), Professor Mamadou Sangharé (President of the African Institute for Mathematical Sciences) and Dr. Long Do Cao (Tutor at AIMS SENEGAL) will wonder around the following question: "The next Einstein will he African?" on RFI see more….

Calendar

<< < May 2014 > >>

S	M	T	W	T	F	S
				1	2	3
4	5	6	7	8	9	10
11	12	13	14	15	16	17
18	19	20	21	22	23	24
25	26	27	28	29	30	31

Spectral insights into shape optimization



Evans Harrell

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May, 2015

What do eigenvalues tell us about shapes?

- M. Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly*, 1966.
  - $-\Delta u = (\omega/c)^2 u =: \lambda u.$

A recent trend in “data mining”

- “Sonification.” Turn data into sound and have people listen to it. The ear is very quick to pick out patterns. We understand much about a shape from the sounds it makes, but what we understand by ear isn’t always the same as what we understand by eye.

Well, can one hear the shape of a drum?

Well, can one hear the shape of a drum?

Gordon, Webb, and Wolpert, 1991

Some things are "audible"

- You can hear the area of the drum, by the *Weyl asymptotics*:
- For the drum problem

$$\lambda_k \sim C_d \,(\mathrm{Vol}(\Omega)/k)^{2/d}.$$

(A mathematician's drum can be d-dimensional, and even a curved manifold.)

Some things are "audible"

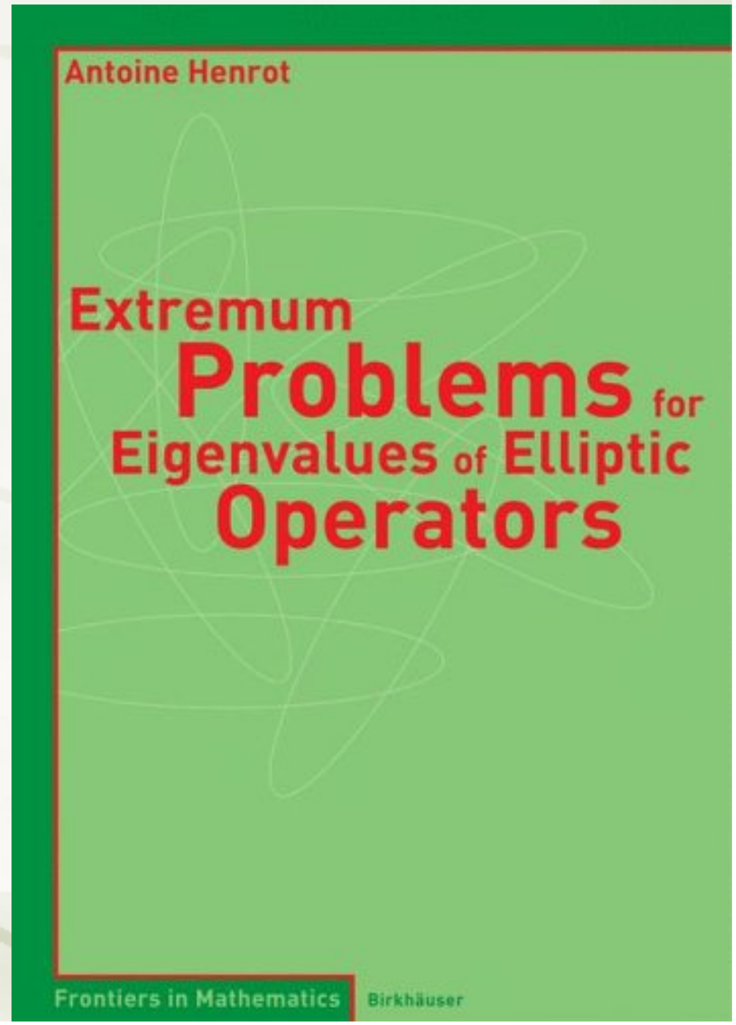
- You can hear the area of the drum, by the *Weyl asymptotics*:
- For the drum problem

$$\lambda_k \sim C_d\,(\mathrm{Vol}(\Omega)/k)^{2/d}.$$

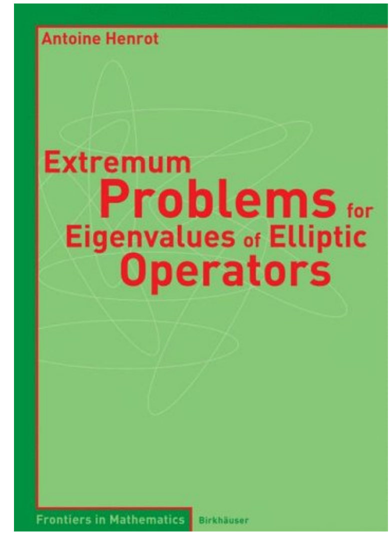
- Notice that in addition to the volume,we can hear the dimension.

You can often “hear” extreme cases

To extremists,
things tend to sound simple…



Is the extremum always a union of round shapes?



No	Optimal union of discs		Computed shapes	
3		46.125		46.125
4		64.293		64.293
5		82.462		**78.47**
6		92.250		**88.96**
7		110.42		**107.47**
8		127.88		**119.9**
9		138.37		**133.52**
10		154.62		**143.45**

Universal inequalities

- They hold for all operators of a given type (say, Laplacians on a domain), and we can ask for the optimal shapes for those inequalities and for best constants.

"Universal" constraints on the spectrum

- H. Weyl, 1910, Laplace, $\lambda_n \sim n^{2/d}$
- W. Kuhn, F. Reiche, W. Thomas, W. Heisenberg, 1925, "sum rules" for atomic energies.
- L. Payne, G. Pólya, H. Weinberger, 1956: The gap is controlled by the average of the smaller eigenvalues:

$$\lambda_{n+1} - \lambda_n \leq \frac{4}{d}\frac{1}{n}\sum_{k\leq n}\lambda_k$$

"Universal" constraints on the spectrum with phase-space volume.

- Lieb -Thirring, 1977, for Schrödinger

$$\epsilon^{\mathbf{d/2}} \sum_{\lambda_{\mathbf{j}}(\epsilon)<\mathbf{0}} |\lambda_{\mathbf{j}}(\epsilon)|^{\rho} \leq \mathbf{L}_{\rho,\mathbf{d}} \int_{\mathbf{R^d}} (\mathbf{V}_{-}(\mathbf{x}))^{\rho+\mathbf{d/2}}\, \mathbf{dx}$$

- Li - Yau, 1983 (Berezin 1973), for Laplace

$$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}$$

- Bounds on ratios (Harrell-Hermi) of averages (k>j)

$$\frac{\overline{\lambda_k}}{\overline{\lambda_j}} \leq \frac{4+d}{2+d} \left(\frac{k}{j}\right)^{2/d}$$

"Universal" constraints on the spectrum

- Ashbaugh-Benguria 1991, isoperimetric conjecture of PPW proved.
- H. Yang 1991, unpublished, formulae like PPW, respecting Weyl asymptotics for the first time.
- Harrell 1993-present, commutator approach; with Michel, Stubbe, El Soufi and Ilias, Hermi, Yildirim.
- Ashbaugh-Hermi, Levitin-Parnovsky, Cheng-Yang, Cheng-Chen, some others.

Two strategies for obtaining universal inequalities and finding cases of optimum

1. Algebraic methods based on commutators of operators.

   Cases of optimality are approached by seeing which ones produce simple relations among commutators.

Two strategies for obtaining universal inequalities and finding cases of optimum

1. Algebraic methods based on commutators of operators.

2. A new variational principle involving averaging.

   Cases of optimality are obtained by microlocal or semiclassical techniques

Commutators of operators

- [G, [H, G]] = 2 GHG - G^2H - HG^2
- Etc., etc. Typical consequence:

$$\langle \phi_\mathbf{j}, [\mathbf{G}, [\mathbf{H}, \mathbf{G}]]\phi_\mathbf{j} \rangle = \sum_{\mathbf{k}:\lambda_\mathbf{k} \neq \lambda_\mathbf{j}} (\lambda_\mathbf{k} - \lambda_\mathbf{j}) |\mathbf{G}_\mathbf{kj}|^\mathbf{2}$$

(Abstract version of Hans Bethe's sum rule from ~1930)

A “sum rule” identity for H=H, G=G*, If J = {λ_1, …, λ_k} ≤ z ≤ J^c:*

$$\frac{1}{2}\sum_{\lambda_j \in J}(z-\lambda_j)^2\,\langle[G,[H,G]]\phi_j,\phi_j\rangle - \sum_{\lambda_j \in J}(z-\lambda_j)\|[H,G]\phi_j\|^2$$

$$=$$

$$\sum_{\lambda_j \in J}\sum_{\lambda_k \in J^c}(z-\lambda_j)(z-\lambda_k)(\lambda_k-\lambda_j)|\,\langle G\phi_j,\phi_k\rangle\,|^2$$

Harrell-Stubbe TAMS 1997

For J = {λ_1…λ_n, the right side ≤0!

Submanifolds (arbitrary codimension) - with El Soufi & Ilias

Theorem 2.1 *Let $X : M \longrightarrow \mathbb{R}^m$ be an isometric immersion. We denote by h the mean curvature vector field of X (i.e the trace of its second fundamental form). For any bounded potential q on M, the spectrum of $H = -\Delta + q$ (with Dirichlet boundary conditions if $\partial M \neq \emptyset$) must satisfy, $\forall k \geq 1$,*

$$(I)\ \ n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\,(\lambda_i + \delta_i)$$

$$\delta_i := \int_M \left(\frac{|h|^2}{4} - q \right) u_i^2$$

Submanifolds - Result is optimal

$$n\sum_{i=1}^{k}\left(\lambda_{k+1}^{sphere}-\lambda_{i}^{sphere}\right)^{2}=\sum_{i=1}^{k}\left(\lambda_{k+1}^{sphere}-\lambda_{i}^{sphere}\right)\left(4\lambda_{i}^{sphere}+n^{2}\right)$$

Statistics of spectra

$$\left(\left(1+\frac{2}{d}\right)\overline{\lambda_k}\right)^2-\left(1+\frac{4}{d}\right)\overline{\lambda_k^2}\geq 0.$$

A *reverse* Cauchy inequality:

The variance is dominated by the square of the mean.

Averaging and spectral inequalities

The dimension of information in a graph

$$\sum_{j=1}^{k-1} \lambda_j \leq \frac{\pi^2|\mathcal{E}|}{3} \left(\frac{k}{|\mathcal{V}|} \right)^{1+\frac{2}{\nu}}$$

Harrell-Stubbe, to appear in *Linear Algebra and Applications*


Can one hear the shape of
data?

There are 74 full wikipedia pages on PDE, but they are not all connected!

The “adjacency matrix” for PDEs in wikipedia

```
PDEAdj = {{0, 1, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
   1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {1, 1, 1, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, 0, 1, 0, 1,
   0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0},
  {0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
   0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1},
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   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1,
   0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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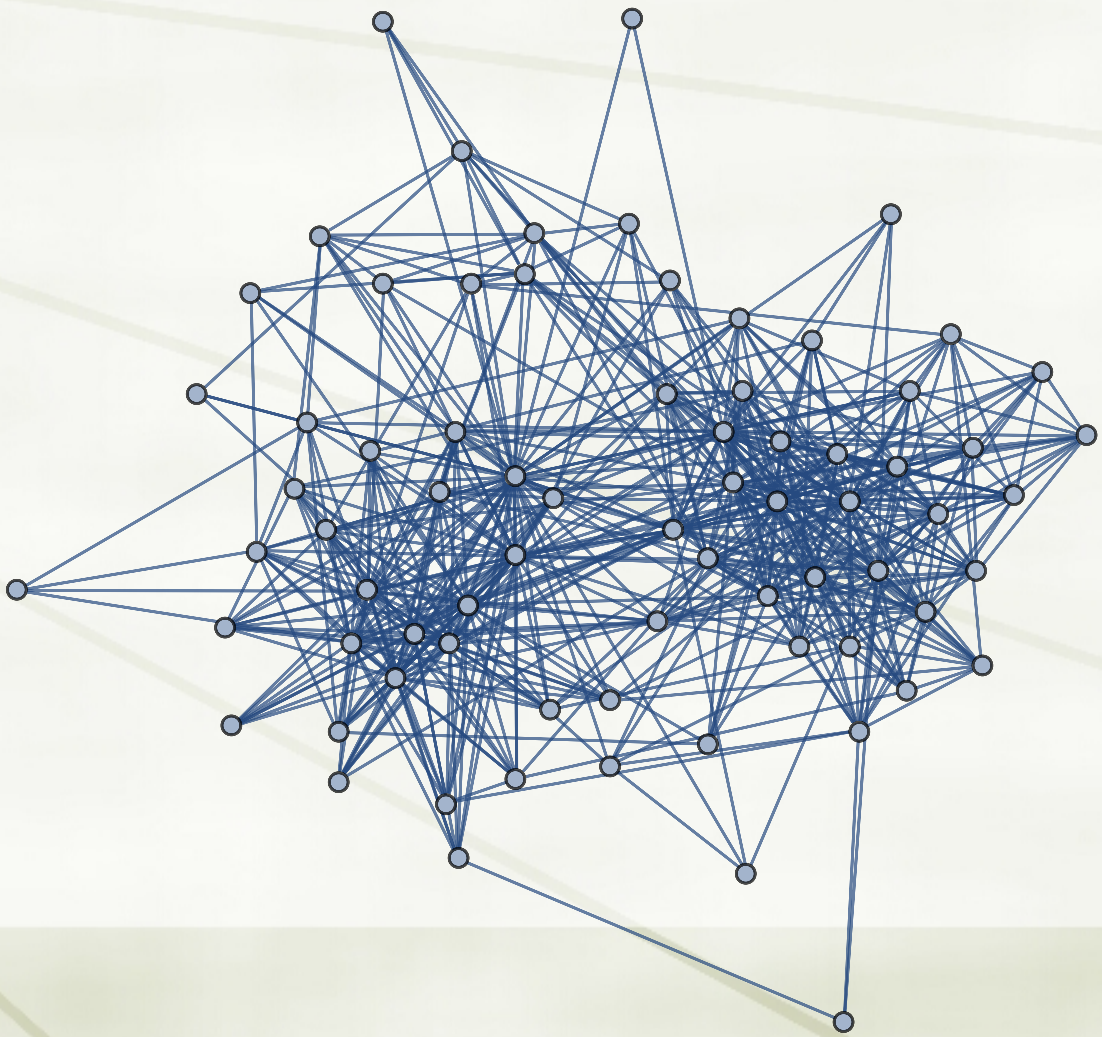
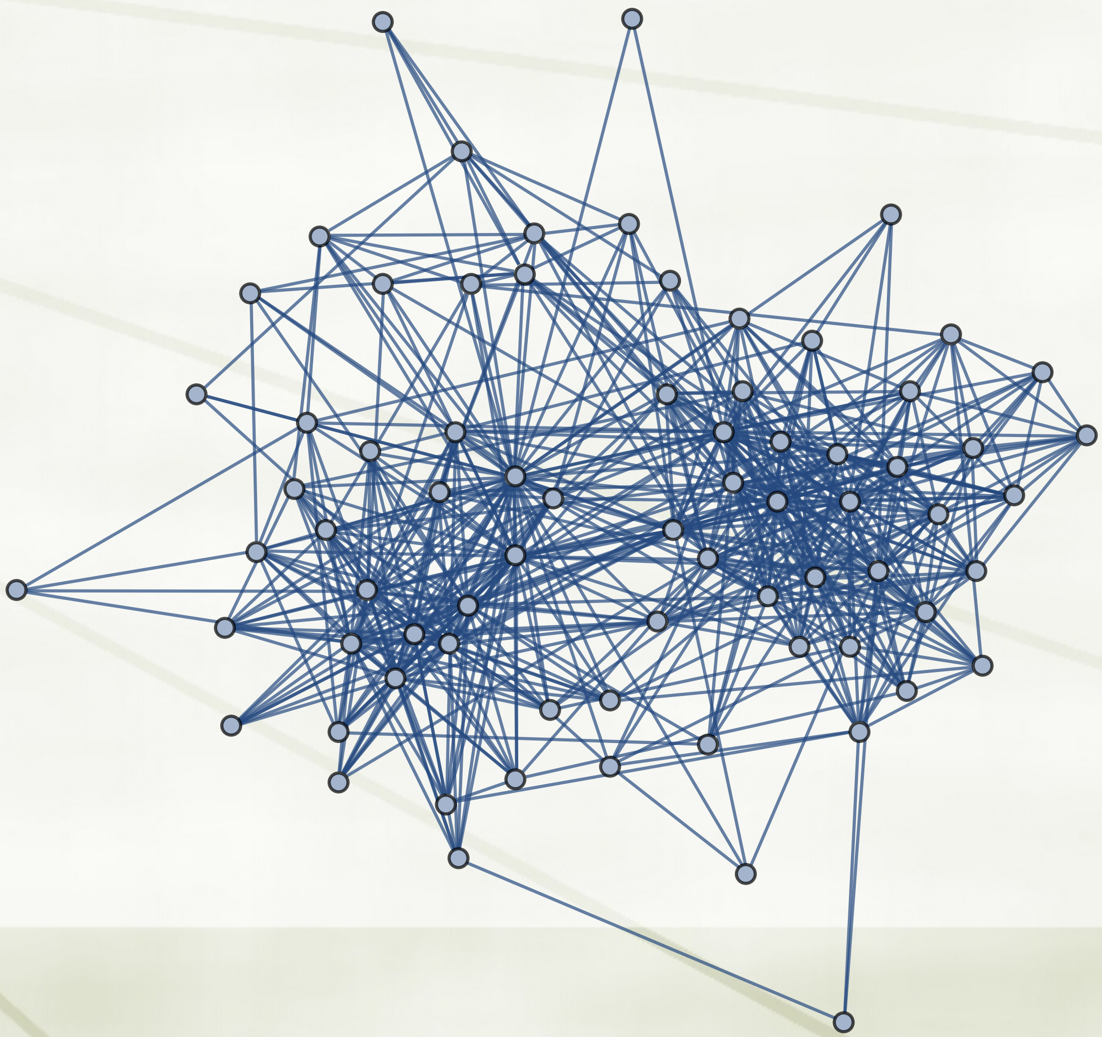
The “adjacency matrix” for PDEs in wikipedia



The distribution of eigenvalues of the ‘graph Laplacian” for this graph indicates that it is dominantly 3 dimensional.



A new tool: an averaged variational principle for sums

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \leq \frac{1}{|\mathfrak{M}_0|}\int_{\mathfrak{M}_0}\frac{Q_M(f_\zeta, f_\zeta)}{\|f_\zeta\|^2}d\sigma.$$

Variational bounds on graph spectra

- In 1992 Pawel Kröger found a variational argument for the Neumann counterpart to Berezin-Li-Yau, i.e. a Weyl-sharp upper bounds on sums of the eigenvalues of the Neumann Laplacian.

- BLY: $$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}$$
- Kröger: $$\sum_{j=0}^{k-1} \mu_j \leq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}$$

The weak form of PDEs with Neumann BC

We can find sharp upper bounds for sums of eigenvalues of expressions defined in a variational quadratic form as follows:

$$\mathcal{E}(\varphi) := \frac{\int_\Omega (|\nabla\varphi(\mathbf{x})|^2 + V(\mathbf{x})|\varphi(\mathbf{x})|^2) w(\mathbf{x}) e^{-2\rho(\mathbf{x})} dv_g}{\int_\Omega |\varphi(\mathbf{x})|^2 e^{-2\rho(\mathbf{x})} dv_g},$$

Where Ω is a domain in a homogeneous space, which has been conformally transformed in an arbitrary way. Weak Neumann conditions correspond to test functions in the restriction of $H_0^1(R^d)$ to Ω. (Evans and Edmunds)

Theorem 1.2. *Let $\mu_0 \leq \mu_1 \leq \ldots$ be the variationally defined Neumann eigenvalues (3) on a bounded open set $\Omega \subset \mathbb{R}^\nu$ endowed with the standard Euclidean metric, where w, ρ, and V satisfy the assumptions stated above. Then*

$$\frac{1}{k}\sum_{j=0}^{k-1} \mu_j \leq \frac{4\pi^2\nu}{\nu+2}\left(\frac{k}{|\Omega|\omega_\nu}\right)^{\frac{2}{\nu}} ⨍_\Omega w(\mathbf{x})d^\nu x + ⨍_\Omega \widetilde{V}(\mathbf{x})w(\mathbf{x})d^\nu x, \tag{9}$$

where $\widetilde{V}(\mathbf{x}) := V(\mathbf{x}) + |\nabla\rho|^2(\mathbf{x})$ and, for every $f \in L^1(\Omega)$, $⨍_\Omega f(\mathbf{x})d^\nu x = \frac{1}{|\Omega|}\int_\Omega f(\mathbf{x})d^\nu x$ is the mean value of f with respect to Lebesgue measure.

Theorem 3.1. *Let (M, g) be a Riemannian manifold of dimension $\nu \geq 2$. Let $\mu_l = \mu_l(\Omega, g, \rho, w, V)$, $l \in \mathbb{N}$, be the eigenvalues defined by (2) and (3) on a bounded open set $\Omega \subset M$, where w, ρ, and V satisfy the assumptions stated above. Then*

(1) For all $z \in \mathbb{R}$,

$$\sum_{j \geq 0} (z - \mu_j)_+ \geq \frac{2\ |\Omega|_g}{(\nu + 2) H_\Omega} \left(\fint_\Omega w \, dv_g \right)^{-\frac{\nu}{2}} \left(z - \fint_\Omega \widetilde{V} w \, dv_g \right)_+^{1+\frac{\nu}{2}} \tag{20}$$

where $\widetilde{V} = V + |\nabla^g \rho|^2$.

(2) For all $k \in \mathbb{N}$,

$$\frac{1}{k} \sum_{j=0}^{k-1} \mu_j \leq \frac{\nu}{\nu + 2} \left(\frac{H_\Omega}{|\Omega|_g} k \right)^{\frac{2}{\nu}} \fint_\Omega w \, dv_g + \fint_\Omega \widetilde{V} w \, dv_g. \tag{21}$$

An averaged variational principle for sums

Theorem 3.1 *Consider a self-adjoint operator M on a Hilbert space $\mathcal{H}$, with ordered, entirely discrete spectrum $-\infty < \mu_0 \leq \mu_1 \leq \dots$ and corresponding normalized eigenvectors $\{\psi^{(\ell)}\}$. Let f_ζ be a family of vectors in $\mathcal{Q}(M)$ indexed by a variable ζ ranging over a measure space $(\mathfrak{M}, \Sigma, \sigma)$. Suppose that $\mathfrak{M}_0$ is a subset of $\mathfrak{M}$. Then for any eigenvalue μ_k of M,*

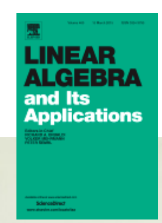
$$\mu_k\left(\int_{\mathfrak{M}_0} \langle f_\zeta, f_\zeta \rangle \, d\sigma - \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma \right)$$

$$\leq \tag{3.2}$$

$$\int_{\mathfrak{M}_0} \langle H f_\zeta, f_\zeta \rangle d\sigma - \sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma,$$

provided that the integrals converge.

Harrell-Stubbe LAA, 2014



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$$
\begin{aligned}
&\mu_k \left(\int_{\mathfrak{M}_0} \langle f_\zeta, f_\zeta \rangle \, d\sigma - \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma \right) \\
&\leq \\
&\int_{\mathfrak{M}_0} \langle H f_\zeta, f_\zeta \rangle d\sigma - \sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma,
\end{aligned}
\tag{3.2}
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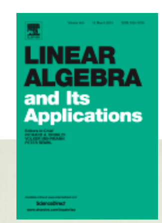

Proof

$$\mu_k\Big(\langle f,f\rangle - \langle P_{k-1}f, P_{k-1}f\rangle\Big) \le \langle Mf,f\rangle - \langle MP_{k-1}f, P_{k-1}f\rangle. \qquad (3.1)$$

Proof. By integrating (3.1),

$$\mu_k \int_{\mathfrak{M}_0} (\langle f_\zeta, f_\zeta\rangle - \langle P_{k-1}f, P_{k-1}f_\zeta\rangle)\, d\sigma \tag{3.3}$$
$$\leq \int_{\mathfrak{M}_0} \langle Mf_\zeta, f_\zeta\rangle\, d\sigma - \int_{\mathfrak{M}_0} \langle MP_{k-1}f_\zeta, P_{k-1}f_\zeta\rangle\, d\sigma,$$

or

$$\mu_k \int_{\mathfrak{M}_0} \left(\langle f_\zeta, f_\zeta\rangle - \sum_{j=0}^{k-1} |\langle f_\zeta, \psi^{(j)}\rangle|^2\right) d\sigma \tag{3.4}$$
$$\leq \int_{\mathfrak{M}_0} \langle Mf_\zeta, f_\zeta\rangle\, d\sigma - \int_{\mathfrak{M}_0} \sum_{j=0}^{k-1} \mu_j |\langle f_\zeta, \psi^{(j)}\rangle|^2\, d\sigma.$$

Since μ_k is larger than or equal to any weighted average of $\mu_1 \ldots \mu_{k-1}$, we add to (3.4) the inequality

$$-\mu_k \int_{\mathfrak{M}\setminus\mathfrak{M}_0} \left(\sum_{j=0}^{k-1} |\langle f_\zeta, \psi^{(j)}\rangle|^2\right) d\sigma \leq -\int_{\mathfrak{M}\setminus\mathfrak{M}_0} \sum_{j=0}^{k-1} \mu_j |\langle f_\zeta, \psi^{(j)}\rangle|^2\, d\sigma, \tag{3.5}$$

and obtain the claim. □

How to use the averaged variational principle to get sharp results?

$$\mu_k\left(\int_{\mathfrak{M}_0} \langle f_\zeta, f_\zeta \rangle \, d\sigma - \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma \right)$$

$$\leq$$

$$\int_{\mathfrak{M}_0} \langle H f_\zeta, f_\zeta \rangle d\sigma - \sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma,$$

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$$\mu_k \cancel{\left(\int_{\mathfrak{M}_0} \langle f_\zeta, f_\zeta \rangle d\sigma - \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma \right)}$$

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How to use the averaged variational principle to get sharp results?

Ans: If $\mathfrak{M}_0$ is large enough that

$$\int_{\mathfrak{M}_0} \langle f_\zeta, f_\zeta \rangle \, d\sigma \geq \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma$$

then

$$\sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma \leq \int_{\mathfrak{M}_0} \langle M f_\zeta, f_\zeta \rangle d\sigma$$


THE END