Sum rules and semiclassical limits for quantum Hamiltonians on surfaces, periodic structures, and graphs.

Evans Harrell
Georgia Tech
www.math.gatech.edu/~harrell

Mathematical aspects of quantum transport and applications in nanophysics
Aalborg, Denmark, 12 August, 2009
Happy 122nd Birthday, Erwin!
Four models suggested by problems in nanophysics

1. Schrödinger operators on curves and surfaces embedded in space. *Quantum wires and waveguides.*


3. Quantum graphs. *Nanoscale circuits*

Four models suggested by problems in nanophysics

1. Schrödinger operators on curves and surfaces embedded in space. *Quantum wires and waveguides.*


3. Quantum graphs. *Nanoscale circuits*

Are the spectra of these models controlled by “sum rules,” like those known for Laplace/Schrödinger on domains or all of $R^d$, or are there important differences?
Stubbe’s proof of sharp Lieb-Thirring for $\rho \geq 2$ (JEMS, in press)
Stubbe’s proof of sharp Lieb-Thirring for \( \rho \geq 2 \) (JEMS, in press)

1. A trace formula ("sum rule") of Harrell-Stubbe '97, for \( H = -\varepsilon \Delta + V \):

\[
R_\rho(z) := \sum (z - \lambda_k)^\rho_+;
\]

\[
R_\rho(z) - \varepsilon \frac{2^\rho}{d} \sum (z - \lambda_k)^\rho_+^{-1} \|\nabla \phi_k\|^2 = \text{explicit expr} \leq 0.
\]
Stubbe’s proof of sharp Lieb-Thirring for $\rho \geq 2$  \( (JEMS, \text{in press}) \)

1. A trace formula (“sum rule”) of Harrell-Stubbe ‘97, for $H = -\varepsilon \Delta + V$:

$$R_\rho(z) := \sum (z - \lambda_k)^\rho_+;$$

$$R_\rho(z) - \varepsilon \frac{2\rho}{d} \sum (z - \lambda_k)^{\rho-1}_+ \| \nabla \phi_k \|^2 = \text{explicit expr} \leq 0.$$

2. $T_k := \langle \phi_k, -\Delta \phi_k \rangle = \frac{d\lambda_k}{d\varepsilon}$ (Feynman-Hellman)
Lieb-Thirring inequalities

Thus

\[ R_\rho(z, \epsilon) \leq - \frac{2\epsilon}{d} \frac{\partial R_\rho(z, \rho)}{\partial \rho}, \]

or:

\[ \frac{\partial}{\partial \rho} \left( \epsilon \frac{d}{2} R_\rho(z, \epsilon) \right) \leq 0, \]

And classical Lieb-Thirring is an immediate consequence!

Recall:

\[
\lim_{\alpha \to 0^+} \alpha^{\frac{d}{2}} \sum_{\lambda_j(\alpha) < 0} |\lambda_j(\alpha)|^\sigma = L_{\sigma,d} \int |V_-(x)|^{\sigma + \frac{d}{2}}
\]
Universal bounds for Dirichlet Laplacians

Payne-Pólya-Weinberger, 1956:

\[
\lambda_{k+1} - \lambda_k \leq \frac{4}{d} \frac{1}{k} \sum_{j=1}^{k} \lambda_j =: \frac{4}{d} \bar{\lambda}_k
\]

Hile-Protter 1980:

\[
1 \leq \frac{4}{d} \frac{1}{k} \sum_{j=1}^{k} \frac{\lambda_j}{\lambda_{k+1} - \lambda_j}
\]

Yang 1991:

\[
\sum_{j=1}^{k} \left( \lambda_{k+1} - \lambda_j \right)^2 \leq \frac{4}{d} \sum_{j=1}^{k} \lambda_j \left( \lambda_{k+1} - \lambda_j \right)
\]
Commutators of operators

- \([H, G] := HG - GH\)
- \([H, G] u_k = (H - \lambda_k) G u_k\)
- If \(H = H^*\),
  \[
  \langle u_j, [H, G] u_k \rangle = (\lambda_j - \lambda_k) \langle u_j, Gu_k \rangle
  \]
Commutators of operators

\[ [G, [H, G]] = 2 \, GHG - G^2H - HG^2 \]

Etc., etc. Typical consequence:

\[ \langle u_j, [G, [H, G]] u_j \rangle = \sum_{k: \lambda_k \neq \lambda_j} (\lambda_k - \lambda_j) |G_{j,k}|^2 \]

(Abstract version of Bethe’s sum rule)
The only assumptions are that $H$ and $G$ are self-adjoint, and that the eigenfunctions are a complete orthonormal sequence. (If continuous spectrum, need a spectral integral on right.)
Or even without G=G*:

\[
\frac{1}{2} \sum_{\lambda_j \in J} (z - \lambda_j)^2 \left( \langle [G^*, [H, G]] \phi_j, \phi_j \rangle + \langle [G, [H, G^*]] \phi_j, \phi_j \rangle \right) \\
- \sum_{\lambda_j \in J} (z - \lambda_j) \left( \langle [H, G] \phi_j, [H, G] \phi_j \rangle + \langle [H, G^*] \phi_j, [H, G^*] \phi_j \rangle \right) \\
= \\
\sum_{\lambda_j \in J} \sum_{\lambda_k \notin J} (z - \lambda_j)(z - \lambda_k)(\lambda_k - \lambda_j) \left( |\langle G \phi_j, \phi_k \rangle|^2 + |\langle G^* \phi_j, \phi_k \rangle|^2 \right),
\]
What you should remember about trace formulae/sum rules in a short seminar?
What you should remember about trace formulae/sum rules in a short seminar?

1. There is an exact identity involving traces including \([G, [H, G]]\) and \([H,G]^*[H,G]\).

2. For the lower part of the spectrum it implies an inequality of the form:

\[
\sum (z - \lambda_k)^2 (...) \leq \sum (z - \lambda_k) (...) 
\]
**Dirichlet problem:**

Trace identities imply differential inequalities

\[ R_2(z) \leq \frac{4}{d} \sum_k (z - \lambda_k) T_k \]

Harrell-Hermi JFA 08: Laplacian

\[
\left(1 + \frac{4}{d}\right) R_2(z) - \frac{2z}{d} R'_2(z) \leq 0.
\]

Consequences – universal bound for \( k > j \):

\[
\frac{\lambda_k}{\lambda_j} \leq \frac{4 + d}{2 + d} \left( \frac{k}{j} \right)^{2/d}
\]
A reverse Cauchy inequality:

\[ \left( \left( 1 + \frac{2}{d} \right) \bar{\lambda}_k \right)^2 - \left( 1 + \frac{4}{d} \right) \bar{\lambda}_k^2 \geq 0. \]

The variance is dominated by the square of the mean.
Statistics of spectra

\[ D_k := \left( \left( 1 + \frac{2}{d} \right) \bar{\lambda}_k \right)^2 - \left( 1 + \frac{4}{d} \right) \bar{\lambda}_k^2 \geq 0. \]

\[ \lambda_{k+1} \leq \left( 1 + \frac{2}{d} \right) \bar{\lambda}_k + \sqrt{D_k}. \]

\[ \lambda_{k+1} - \lambda_k \leq 2\sqrt{D_k}. \]
The models: What do LT and universal bounds look like?

1. Quantum graphs. *Nanoscale circuits*

Quantum graphs

Work in progress with S. Demirel, Stuttgart. For which graphs is:

\[ R_\sigma(z) = \sum (z - \lambda_k) \rho + \quad R_\rho(z) - \alpha^2 \rho \sum (z - \lambda_k) \rho - 1 + \|\nabla \phi_k\|^2 = \text{explicit expr} \leq 0. \]

\[ \lambda_{k+1} \leq (1 + 2d) \lambda_k + \sqrt{D_k} \]

\[ R_\sigma(z) \leq L_{cl}^{\sigma,1} \int_\Gamma (V(x) - z)^{\sigma+1/2} dx? \]

(Concentrate on \( \sigma=2 \).)
Quantum graphs

1. Trees.
Quantum graphs

1. Trees.

2. Scottish tartans (infinite rectangular graphs):

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Quantum graphs

1. Trees.
2. Infinite rectangular graphs.
3. Bathroom tiles, a.k.a. honeycombs, etc.:
Quantum graphs

1. But not balloons! (A.k.a. tadpoles, or...)
Put soliton potential on the loop.

\[ V(\chi) = \frac{-2a^2}{\cosh^2(ax)} \chi \quad \text{on } \partial \Omega \]

\[ \phi(s) = \int_{s}^{\infty} e^{-ax} \frac{\cosh(\alpha L)}{\cosh(\alpha x_0)} \, dx \]

\[ \lambda_1 = -a^2 \quad \text{solves a transcendental eqn,} \]

but \[ \left| \lambda_1 \right| \frac{6}{\int_{1/2}^{6+1/2} \, dx} \quad \text{can be determined exactly!} \]
Quantum graphs

1. But not balloons! (A.k.a. tadpoles, or...)

\[ G = \frac{3}{2} : \quad \frac{3}{11} \quad \text{vs} \quad L_{\frac{3}{2},1}^{cl} = \frac{3}{16} \]

\[ G = 2 : \quad \text{messy expression} = 0.20092 \]

\[ \text{vs} \quad L_{\frac{8}{21}}^{cl} = \frac{8}{15\pi} = 0.169765 \]
Quantum graphs

For which finite graphs is:

e.g., is $\lambda_2/\lambda_1 \leq 5$?

\[
\frac{x_k}{x_i} \leq 4 + \frac{d}{2} \frac{(k)}{(j)} \frac{2/d}{(1)}
\]

\[1 = 4 \sum_{k} \frac{\lambda_k}{\lambda_i} \frac{|\langle \varphi_j, \nabla \varphi_k \rangle|^2}{\lambda_k - \lambda_j} \alpha R_2(0)\]
Quantum graphs

1. Trees.
Quantum graphs

1. Trees.
2. Rectangular graphs/bathroom tiles with external edges:
Quantum graphs

• But not balloons!
Quantum graphs

- Fancy balloons can have arbitrarily large $\lambda_2/\lambda_1$. 
Why?
If we can establish the analogue of the trace inequality,

\[
R_{\rho}(z) - \alpha \frac{2\rho}{d} \sum (z - \lambda_k)^{\rho-1} \| \nabla \phi_k \|^2 \leq 0,
\]

then all the rest of the inequalities follow (LT, PPW, ratios, statistics, etc.), sometimes with modifications.
In each of the four models there are new features in the trace inequality.

1. Schrödinger operators on curves and surfaces. *Explicit curvature terms.*

2. Periodic Schrödinger operators. *Geometry of the dual lattice.*

3. Quantum graphs. *Topology*

4. Relativistic Hamiltonians. *First-order \( \Psi DO \) rather than second-order.*
1st and 2nd commutators (H-S ’97)

\[ \frac{1}{2} \sum_{\lambda_j \in J} (z - \lambda_j)^2 \langle [G, [H, G]] \phi_j, \phi_j \rangle - \sum_{\lambda_j \in J} (z - \lambda_j) \| [H, G] \phi_j \|^2 = \]

\[ \sum_{\lambda_j \in J} \sum_{\lambda_k \in J^c} (z - \lambda_j)(z - \lambda_k)(\lambda_k - \lambda_j) \| \langle G \phi_j, \phi_k \rangle \|^2 \]
Suppose $G'$ is constant on each edge and $\phi \rightarrow G\phi$ preserves vertex conditions. (In the case this means $G$ continuous, $\sum \frac{\partial G}{\partial x_i} = 0$ at $V$.)

The sum rule becomes

$$\sum_j \sum_k \left( (z-\lambda_k)^2 \left( a_j \|\phi_k\|_{\Pi_j}^2 - 4\alpha (z-\lambda_j)^2 a_j \|\phi_k\|_{\Pi_j}^2 \right) \right)$$

$$= \ldots \leq 0$$

where $|G'(z)|^2 = a_j$ on $\Pi_j \subset \Pi$. 
\[
\sum_{k \leq n} \sum_{j} a_{ij} \left( (z - \lambda_k)^2 \| \phi_k \|^2 - 4x(z - \lambda_k) \| \phi_k \| \| \phi_j \| \right) \leq 0
\]

* Can we find a family of such G's and sum, so that we get

\[
\sum_{k \leq n} \left( (z - \lambda_k)^2 - 4x(z - \lambda_k) \| \phi_k \|^2 \right) \leq 0?
\]
Example. \( G = \text{distance along a "transit" in a tree, constant elsewhere.} \)

\[ a_j = 1 \quad \text{and} \quad a_j = 0 \]

Example. Scottish tantan, \( G = x + y \)

\[ a = 1 \quad \text{(good).} \]
Sharpest inequalities if $c_j$ always 1 and, equally good, $\sum \frac{c_j}{a_j(x)} \chi_{\mathbf{r}_j} = 1$.

Then:

$$R_2(z) := \sum_j (z - \lambda_j)_+^2 \leq 4 \times \sum_j (z - \lambda_j)_+^1 T_j$$

$$R_6(z) \leq 26 \times \sum_j (z - \lambda_j)_+^1 T_j \quad 6 \geq 2$$

$$\leq 4 \sum_j (z - \lambda_j)_+^1 T_j \quad 1 \leq 6 \leq 2$$
Commutation fn loops

Use non-self-adjoint trace formula with \( G = e^{i4x_1} \), \( \theta = \frac{2\pi}{L} \), and extend by a constant on exterior parts.

\[ T_j \rightarrow T_j + \frac{L^2}{4} \]

systematically weakening the inequality
Klein–Gordon and Heaviside operators.

Q1. How do we define $\sqrt{p^2 + m^2}$ in the presence of boundaries?

As Probabilists have an answer—think of random processes with killing condition at boundary.
A’. Use Fourier transform:

\[ \phi \rightarrow \hat{\phi}(k) \rightarrow \sqrt{k^2 + m^2} \hat{\phi}(k) \]

\[ \rightarrow X_{\omega} F^{-1} \sqrt{k^2 + m^2} \hat{\phi}(k) \]

Because of \( X_{\omega} \), this operator is \( \leq \sqrt{p^2 + m^2} \) spectral term.
Eigenvalue inequalities for Klein-Gordon Operators

Evans M. Harrell II
School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160 U.S.A.

Selma Yıldırım Yolcu
School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160 U.S.A.

Abstract

We consider the pseudodifferential operators $H_{m,\Omega}$ associated by the prescriptions of quantum mechanics to the Klein-Gordon Hamiltonian $\sqrt{|P|^2 + m^2}$ when restricted to a bounded, open domain $\Omega \in \mathbb{R}^d$. When the mass $m$ is 0 the operator $H_{0,\Omega}$ coincides with the generator of the Cauchy stochastic process with a killing condition on $\partial \Omega$. (The operator $H_{0,\Omega}$ is sometimes called the fractional Laplacian with power $\frac{1}{2}$, cf. [15].) We prove several universal inequalities for the eigenvalues $0 < \beta_1 < \beta_2 \leq \cdots$ of $H_{m,\Omega}$ and their means $\bar{\beta}_k := \frac{1}{k} \sum_{\ell=1}^{k} \beta_\ell$. 

Preprint submitted to Elsevier December 2008
Among the inequalities proved are:

\[ \bar{\beta}_k \geq \text{cst.} \left( \frac{k}{|\Omega|} \right)^{1/d} \]

for an explicit, optimal “semiclassical” constant depending only on the dimension \( d \). For any dimension \( d \geq 2 \) and any \( k \),

\[ \beta_{k+1} \leq \frac{d + 1}{d - 1} \bar{\beta}_k. \]

Furthermore, when \( d \geq 2 \) and \( k \geq 2j \),

\[ \frac{\bar{\beta}_k}{\bar{\beta}_j} \leq \frac{d}{2^{1/d}(d-1)} \left( \frac{k}{j} \right)^{\frac{1}{d}}. \]

Finally, we present some analogous estimates allowing for an operator including an external potential energy field, i.e, \( H_{m,\Omega} + V(x) \), for \( V(x) \) in certain function classes.
\[
\sqrt{-\Delta} + m^2 \varphi := \mathcal{F}^{-1} \sqrt{|\xi|^2 + m^2} \hat{\varphi}(\xi).
\]

\[
\exp(-\sqrt{-\Delta} t) [\varphi] (x) = p_0(t, \cdot) \ast \varphi,
\]

\[
p_0(t, x) := \frac{c_d t}{(t^2 + |x|^2)^{\frac{d+1}{2}}},
\]
Comparison to the free Laplacian

\[
\langle \varphi, H_{m,\Omega}^2 \varphi \rangle = \| H_{m,\Omega} \varphi \|^2 = \int_\Omega \left| \mathcal{F}^{-1} \left( \sqrt{\| \xi \|^2 + m^2 \hat{\varphi}} \right) \right|^2 \\
= \int_{\mathbb{R}^d} \left| \chi_\Omega \mathcal{F}^{-1} \left( \sqrt{\| \xi \|^2 + m^2 \hat{\varphi}} \right) \right|^2 \\
\leq \int_{\mathbb{R}^d} \left| \mathcal{F}^{-1} \left( \sqrt{\| \xi \|^2 + m^2 \hat{\varphi}} \right) \right|^2 \\
= \int_{\mathbb{R}^d} \varphi (-\Delta + m^2) \varphi \\
= \int_\Omega \varphi (-\Delta + m^2) \varphi,
\]

Therefore

\[
\beta_k \leq \sqrt{\lambda_k + m^2}.
\]
Calculate first and second commutators:

\[
[H_{m,\Omega}, x_\alpha] \varphi = (H_{m,\Omega} x_\alpha - x_\alpha H_{m,\Omega}) \varphi \\
= \chi_{\Omega} \mathcal{F}^{-1} \sqrt{|\xi|^2 + m^2} \mathcal{F}[x_\alpha \varphi] - \chi_{\Omega} x_\alpha \mathcal{F}^{-1}[\sqrt{|\xi|^2 + m^2} \hat{\varphi}] \\
= \chi_{\Omega} \mathcal{F}^{-1} \left[ \sqrt{|\xi|^2 + m^2} \frac{\partial \hat{\varphi}}{\partial \xi_\alpha} - \frac{\partial}{\partial \xi_\alpha} (\sqrt{|\xi|^2 + m^2} \hat{\varphi}) \right] \\
= -i \chi_{\Omega} \mathcal{F}^{-1} \frac{\xi_\alpha}{\sqrt{|\xi|^2 + m^2}} \hat{\varphi}.
\]

Similarly,

\[
[x_\alpha, [H_{m,\Omega}, x_\alpha]] \varphi = \chi_{\Omega} \mathcal{F}^{-1} \left[ \left( \frac{1}{\sqrt{|\xi|^2 + m^2}} - \frac{\xi_\alpha^2}{(|\xi|^2 + m^2)^{3/2}} \right) \hat{\varphi} \right].
\]
Summing over coordinates:

\[(d - 1) \sum_{j=1}^{n} (z - \beta_j)^2 \langle u_j, H_{m,\Omega}^{-1} u_j \rangle - 2 \sum_{j=1}^{n} (z - \beta_j) \leq 0,\]

provided \(z \in [\beta_n, \beta_{n+1}]\)

or, equivalently,

\[(d - 1)\overline{\beta_n^{-1}}z^2 - 2dz + (d + 1)\overline{\beta_n} \leq 0.\]
\[ \beta_{n+1} \leq \frac{d + 1}{(d - 1)\beta_n^{-1}} \leq \frac{d + 1}{d - 1} \beta_n. \]

In particular,

\[ \frac{\beta_2}{\beta_1} \leq \frac{d + 1}{d - 1}, \]
Corollary 2.4 For $k > 2j$, Eq. (2.24) implies

$$\frac{\beta_k}{\beta_j} \leq \frac{d}{2^{1/d}(d - 1)} \left( \frac{k}{j} \right)^{\frac{1}{d}}.$$
Weyl asymptotic for $H_{\Omega,m}$

Proposition 3.1  As $\beta \to \infty$,

$$N(\beta) \sim \frac{|\Omega|}{(4\pi)^{d/2} \Gamma(1 + d/2)} \beta^d.$$ 

Equivalently, as $k \to \infty$,

$$\beta_k \sim \sqrt{4\pi} \left( \frac{\Gamma(1 + d/2) k}{|\Omega|} \right)^{1/d}.$$
What if $V \neq 0$?
What if V ≠ 0?

(a) If V ≥ 0, then for each k, (2.6) holds. That is

\[
\beta_{k+1} \leq \frac{d+1}{(d-1)\beta_k^{-1}} \leq \frac{d+1}{d-1} \beta_k.
\]

Moreover, for k > 2j, (2.25) holds. That is,

\[
\frac{\bar{\beta}_k}{\bar{\beta}_j} \leq \frac{d}{2^{1/d}(d-1)} \left( \frac{k}{j} \right)^{\frac{1}{d}}.
\]

(b) If V ∈ L^s for some 2 ≤ d < s < ∞, and

\[
\alpha := \frac{\|V\|_s (d-2)! (s-1)^{\frac{s-1}{s}}}{\sqrt{\pi} 2^{(d-1)^2 \frac{1}{d}} \Gamma \left( \frac{d}{2} \right)^{\frac{1-2d}{d}} (d|\Omega|)^{\frac{d-s}{sd}} (s-d)^{\frac{s-1}{s}}} < 1,
\]

then for each k, the eigenvalues β_k satisfy

\[
\frac{\beta_{k+1}}{\beta_k} \leq \frac{1}{\beta_k} \beta_{k+1} \leq 1 + \frac{2}{(d-1)(1-\alpha)}.
\]
Proof. By Karamata’s Tauberian theorem [36], if we can show that for \( t \to 0 \),
\[
t^d Z(t) \to c_d |\Omega|,
\]
then the first claim follows from (3.1). The further claims for \( \beta_k \) and \( U(z) \) are easy consequences of (3.3). By a standard comparison,
\[
p_\Omega(x, y, t) < p_0(x - y, t) \tag{3.5}
\]
on \( \Omega \), where \( p_\Omega \) is the integral kernel of the semigroup \( e^{-tH_{0, \Omega}} \). Define
\[
r_\Omega(x, y, t) := p_0(x - y, t) - p_\Omega(x, y, t),
\]
and let \( \delta_\Omega(x) := \text{dist}(x, \partial \Omega) \). According to [4],
\[
0 \leq r_\Omega(x, y, t) \leq \frac{t}{\delta_\Omega^{d+1}(x)} c_d \mathcal{P}_y(\tau_\Omega < t),
\]
where \( \mathcal{P}_y(\tau_\Omega < t) \) is the probability that a path originating at \( y \) exits \( \Omega \) before time \( t \). From (3.5),
\[
\int_\Omega p_\Omega(x, x, t) dx \leq \int_\Omega p_0(0, t) dx = c_d \frac{|\Omega|}{t^d},
\]
and we proceed to calculate:
\[
\int_\Omega p_\Omega(x, x, t) dx = \int_{\{x : \delta_\Omega(x) < \sqrt{t}\}} p_\Omega(x, x, t) dx + \int_{\{x : \delta_\Omega(x) > \sqrt{t}\}} (p_0(0, t) - r_\Omega(x, x, t)) dx
\]
\[
= \int_{\{x : \delta_\Omega(x) < \sqrt{t}\}} p_\Omega(x, x, t) dx + |\{x : \delta_\Omega(x) > \sqrt{t}\}|c_d t^{-d}
\]
\[
- \int_{\{x : \delta_\Omega(x) > \sqrt{t}\}} r_\Omega(x, x, t)) dx
\]
(3.6)
\[
\int_\Omega p_\Omega(x, x, t)dx = \int_{\{x: \delta_\Omega(x) < \sqrt{t}\}} p_\Omega(x, x, t)dx + \int_{\{x: \delta_\Omega(x) > \sqrt{t}\}} (p_0(0, t) - r_\Omega(x, x, t))dx
\]
\[
= \int_{\{x: \delta_\Omega(x) < \sqrt{t}\}} p_\Omega(x, x, t)dx + \left|\{x: \delta_\Omega(x) > \sqrt{t}\}\right| c_d t^{-d}
\]
\[- \int_{\{x: \delta_\Omega(x) > \sqrt{t}\}} r_\Omega(x, x, t)dx \]
\[
(3.6)
\]

The first integral on the right side of (3.6) becomes
\[
0 \leq \int_{\{x: \delta_\Omega(x) < \sqrt{t}\}} p_\Omega(x, x, t)dx \leq \int_{\{x: \delta_\Omega(x) < \sqrt{t}\}} p_0(0, t)dx
\]
\[
\leq c_d t^{-d} \left|\{x: \delta_\Omega(x) < \sqrt{t}\}\right| = o(t^{-d})
\]
\[
(3.7)
\]
as \(t \to 0\). As for the final integral of (3.6),
\[
0 \leq \int_{\{x: \delta_\Omega(x) > \sqrt{t}\}} r_\Omega(x, x, t)dx \leq \int_{\{x: \delta_\Omega(x) > \sqrt{t}\}} \frac{t}{\delta^{d+1}_\Omega(x)} dx
\]
\[
\leq \frac{t}{t^{(d+1)/2}} |\Omega|
\]
\[
= 0(t^{(1-d)/2}) = o(t^{-d}).
\]
\[
(3.8)
\]
Articles related to this seminar

- S. Demirel and E.M. Harrell, manuscript in prep.
- E.M. Harrell, Commutators, eigenvalue gaps, and mean curvature in the theory of Schrödinger operators, Communications PDE, 2007
- E.M. Harrell and J. Stubbe, Trace identities for eigenvalues, with applications to periodic Schrödinger operators and to the geometry of numbers, preprint 2009.
THE END