Optimal estimates of sums of eigenvalues and heat traces

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Abstract

I'll present two rather distinct results whose common theme is to say something optimal about heat traces. Etc. etc.

Parts of this work are joint with Ahmad El Soufi, Said Ilias, and Joachim Stubbe.

Part 1

A new obstacle problem. (“I'll present two rather distinct results”)

The original spectral isoperimetric theorem

- Lowest eigenvalue of $-\Delta$:
  - Faber-Krahn is the classic result, 1923-5.
- Among all domains of a given volume, the ball is the minimizer of the lowest eigenvalue.

Luttinger's theorem

- Luttinger (1973) looked instead at the *partition function* (physical term) = *heat trace* (more current in math):

$$Z(t) := \mathrm{tr} e^{-tH} = \sum_k e^{-t\lambda_k}$$

- For H = -Δ, DBC on a domain of given volume, he showed that for each t, Z(t) is maximized by the ball.

Faber-Krahn and Luttinger

- Luttinger implies F-K because for large values of t,

$$Z(t) \approx \exp(-t\lambda_1)$$

But by various expansions and transforms it also implies Weyl asymptotics, estimates of the spectral zeta function, regularized determinant, etc.

The obstacle problem

- Fix an outer boundary, and exclude from the region Ω a subset B of a fixed shape (in practice round), but unspecified position.
- How can the extremal values of an eigenvalue or other spectral function be achieved be achieved by moving B around?

The obstacle problem

- Harrell-Kröger-Kurata (2001) analyzed this problem for the ground-state Dirichlet eigenvalue of the Laplacian.
- For some category of regions Ω, the max is achieved when B is in a distinguished subset, and the min is when B touches the boundary.

The obstacle problem

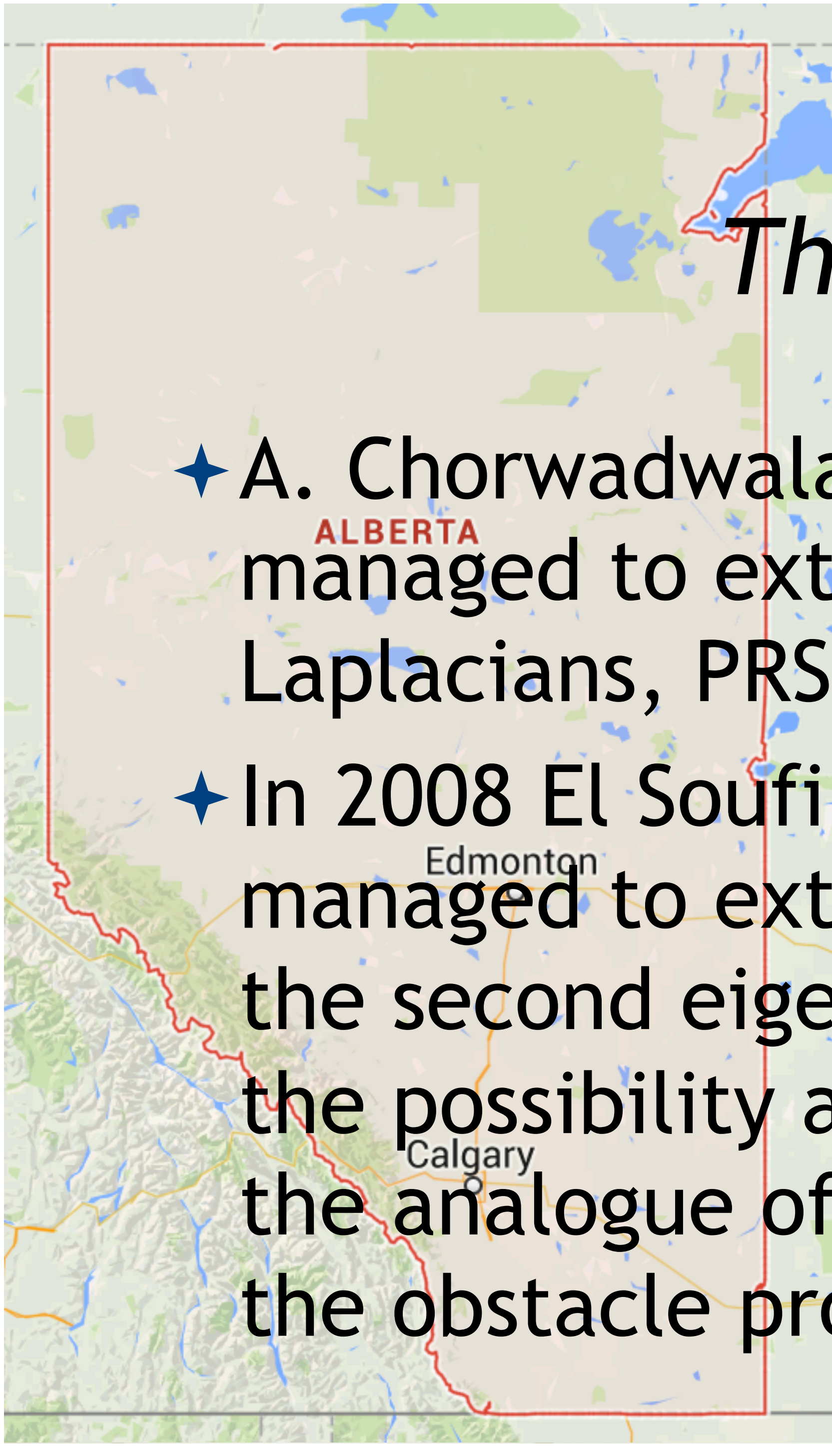
- The model case is an annular region:

Hersch, 1963 for this case. & tahnks to
Antoine for reviewing the later history!

The obstacle problem

The obstacle problem

- The methods of HKK were
  - A moving plane argument
  - Hadamard perturbation formula
  - Maximum principle to establish the sign of the derivative of the eigenvalue when displaced.



The obstacle problem

- A. Chorwadwala and R. Mahadevan managed to extend this result to p-Laplacians, PRSE.
- In 2008 El Soufi and his student Kiwan managed to extend the HKK result to the second eigenvalue, which raised the possibility and challenge of finding the analogue of Luttinger’s result for the obstacle problem.

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The heart of the matter

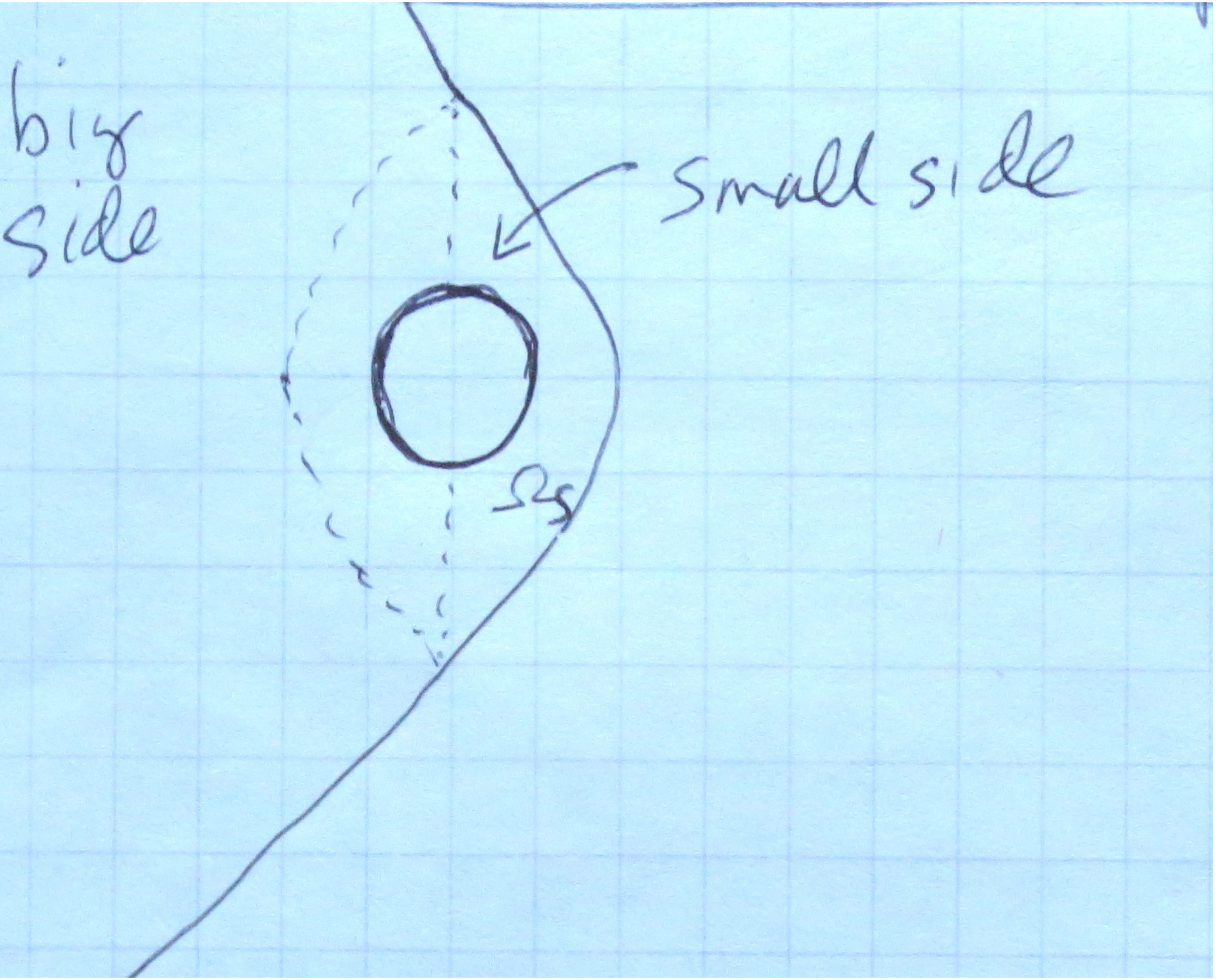
Convex geometers have recently developed a notion of the “heart” of a convex body, which is a distinguished smaller, non-strictly convex set. Actually, the heart can be defined for general domains, but it might be something trivial.

The heart of the matter

Definition 1.2. *In [HKK], the domain D was said to have the* interior reflection property *with respect to a hyperplane P if there is a connected component D_s of $D \setminus P$ whose reflection through P is a proper subset of another connected component of D. Any such P will be called a* hyperplane of interior reflection *for D. The component D_s will be called the* small side *of D (with respect to P) and the other connected component D_b will be called the* big side.

The heart *of D is defined as the relative closure of the set of points $\mathbf{x} \in D$ so that there is no hyperplane of interior reflection passing through $\mathbf{x}$. We denote it $\heartsuit(D)$.*

Brasco, Magnagnini, Salani, ≥ 2011


big
side
small side
Ω_S

Neither Calgary nor Edmonton is in the heart of Alberta.

The heart of the matter

Definition 1.2. *In [HKK], the domain D was said to have the* interior reflection property *with respect to a hyperplane P if there is a connected component D_s of $D \setminus P$ whose reflection through P is a proper subset of another connected component of D. Any such P will be called a* hyperplane of interior reflection *for D. The component D_s will be called the* small side *of D (with respect to P) and the other connected component D_b will be called the* big side.

The heart *of D is defined as the relative closure of the set of points $\mathbf{x} \in D$ so that there is no hyperplane of interior reflection passing through $\mathbf{x}$. We denote it $\heartsuit(D)$.*

The heart of a bounded domain D is a nonempty closed subset of D. Moreover, for a strictly convex bounded D one has $\mathrm{dist}(\heartsuit(D), \partial D) > 0$. We observe that for the ball and for many other domains with sufficient symmetry to identify an unambiguous center, the heart is simply the center.

Step 1. Hadamard formula for Z(t)

- A displacement can be regarded as a boundary perturbation. Ozawa '78, El Soufi-Ilias '07:

$$\frac{\partial}{\partial \varepsilon} Z_{\Omega_\varepsilon}(t)\big|_{\varepsilon=0} = -\frac{t}{2}\int_{\partial\Omega} v\,\Delta K(t,\mathbf{x},\mathbf{x})dx,$$

where $v = X \cdot \nu$ is the component of the deformation vectorfield X in the direction of the inward unit normal ν and $\Delta K(t,\mathbf{x},\mathbf{x})$ stands for the Laplacian of the function $\mathbf{x} \mapsto K(t,\mathbf{x},\mathbf{x})$

Step 2. Reflect the heat kernel through any plane of interior reflection

Define the function $\phi(t, \mathbf{x}, \mathbf{y}) = K(t, \mathbf{x}, \mathbf{y}) - K(t, \mathbf{x}^*, \mathbf{y}^*)$ on $(0, \infty) \times \Omega_s \times \Omega_s$ with $\Omega_s = D_s \setminus B_s$.

Claim : For all $(t, \mathbf{x}, \mathbf{y}) \in (0, \infty) \times \Omega_s \times \Omega_s$, $\phi(t, \mathbf{x}, \mathbf{y}) \leq 0$.
Indeed, for all $\mathbf{x} \in \bar{\Omega}_s$, the function $(t, \mathbf{y}) \mapsto \phi(t, \mathbf{x}, \mathbf{y})$ is a solution of the following parabolic problem :

$$(*) \begin{cases} (\frac{\partial}{\partial t} - \Delta_y)\phi(t, \mathbf{x}, \mathbf{y}) = 0 \ \text{ in } \Omega_s \\ \\ \phi(0^+, \mathbf{x}, \mathbf{y}) = 0. \end{cases}$$

Step 3. Maximum principle for $\phi(x,y)$.

- By checking the signs of $\phi(x,y)$ on the boundary portion of the region we produce by reflecting the boundary inward, we can establish that it has one sign on the interior.

Step 4. Establishing the sign of $\Delta\phi(x,x)$.

- However, what we really need is an analogous result for $\Delta\phi(x,x)$ on the boundary.

Step 4. Establishing the sign of $\Delta\phi$ (x,x).

- We use the formula

$$K(t,\mathbf{x},\mathbf{x}) = \sum_{k\geq 1} e^{-\lambda_k(\Omega)t} u_k(\mathbf{x})^2,$$

to show that it vanishes quadratically at the obstacle, and with some asymptotics and the maximum principle for $\Delta\phi$ (x,y) we get

$$\Delta\phi\,(x,x) \leq 0$$

on the boundary of the obstacle.

Step 4. Establishing the sign of Δϕ (x,x).

Claim : $\Delta\phi(t,\mathbf{x},\mathbf{x}) \le 0$ for all $(t,\mathbf{x}) \in (0,\infty) \times (\partial B)_s$.

As we have seen, the function $\mathbf{x} \in \Omega_s \mapsto \phi(t,\mathbf{x},\mathbf{x})$ achieves its maximum at the boundary. Moreover, since $K(t,\mathbf{x},\mathbf{x}) = \sum_{k\ge 1} e^{-\lambda_k(\Omega)t} u_k(\mathbf{x})^2$, the function $\phi(t,\mathbf{x},\mathbf{x})$ vanishes quadratically on $(\partial B)_s$. Thus, for any $\mathbf{x}_0 \in (\partial B)_s$

$$\nabla\phi(t,\mathbf{x}_0,\mathbf{x}_0) = 0.$$

Taking polar coordinates (ρ,σ) centered at the center of the ball B and writing $\mathbf{x}_0 = (\rho_0,\sigma_0)$ we see that, since all the first derivatives of $\mathbf{x} \in \Omega_s \mapsto \phi(t,\mathbf{x},\mathbf{x})$ vanish at $\mathbf{x}_0$,

$$\Delta\phi(t,\mathbf{x}_0,\mathbf{x}_0) = \frac{\partial^2}{\partial\rho^2}\phi(t,\rho_0,\sigma_0,\rho_0,\sigma_0).$$

This is nonpositive, since $\rho \mapsto \phi(t,\rho,\sigma_0,\rho,\sigma_0)$ achieves its maximum at $\rho = \rho_0$.

Step 5. Analytic function theory.

- With Hadamard, we get

$$\frac{\partial}{\partial \varepsilon} Z_{\Omega_\varepsilon}(t)\Big|_{\varepsilon=0} \geq 0$$

but we need strict positivity at least a.e. We therefore establish that Z and its derivative in ε are analytic in the right half t-plane, and apply unique continuation.

Step 5. Analytic function theory.

- With Hadamard, we get

$$\frac{\partial}{\partial \varepsilon} Z_{\Omega_\varepsilon}(t)\Big|_{\varepsilon=0} \geq 0$$

  but we need strict positivity at least a.e. We therefore establish that Z and its derivative in ε are analytic in the right half t-plane, and apply unique continuation.
- Truthfully, we cannot exclude isolated exceptional t's for which the derivative is 0, but that's enough.

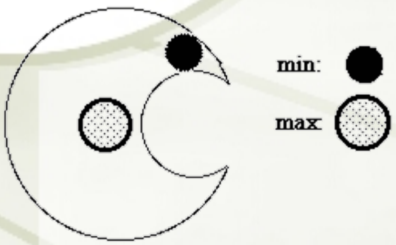
Theorem 1.3. *Let D be a bounded C^2 domain of $\mathbb{R}^n$ and let $r > 0$. For every $\mathbf{x} \in D$ so that the ball $B(\mathbf{x}, r)$ is contained in D, we set $\Omega(\mathbf{x}) = D \setminus B(\mathbf{x}, r)$.*

(i) For every $t > 0$, let $\mathbf{x}_0(t)$ and $\mathbf{x}_1(t)$ be such that the function $\mathbf{x} \mapsto Z_{\Omega(\mathbf{x})}(t)$ achieves its minimum at $\mathbf{x}_0(t)$ and its maximum at $\mathbf{x}_1(t)$. Then $\mathbf{x}_0(t)$ belongs to $\heartsuit_r(D) := \heartsuit(D) \cap \{\mathbf{x} : \mathrm{dist}(\mathbf{x}, \partial D) \geq r\}$ and $\mathbf{x}_1(t)$ is either an interior point of $\heartsuit(D)$ or $\mathrm{dist}(\mathbf{x}_1(t), \partial D) = r$ (i.e. $B(\mathbf{x_1}(\mathbf{t}), r)$ touches the boundary of D).

(ii) Let $\mathbf{x}'_0$ and $\mathbf{x}'_1$ be such that the regularized determinant of the Laplacian $\mathbf{x} \mapsto \det(\Omega(\mathbf{x}))$ achieves its maximum at $\mathbf{x}'_0$ and its minimum at $\mathbf{x}'_1$. Then $\mathbf{x}'_0$ belongs to $\heartsuit_r(D)$ and $\mathbf{x}'_1$ is either an interior point of $\heartsuit(D)$ or $\mathrm{dist}(\mathbf{x}'_1, \partial D) = r$.

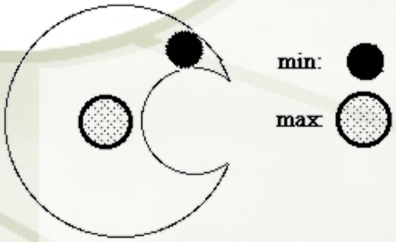
Illustrative examples:

These are pictures from HKK, but just reverse "min" and "max"

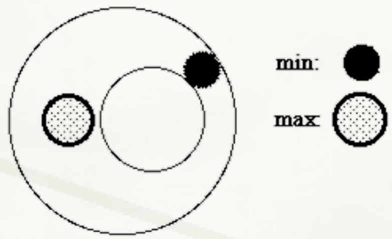


Example 1(b)

Illustrative examples

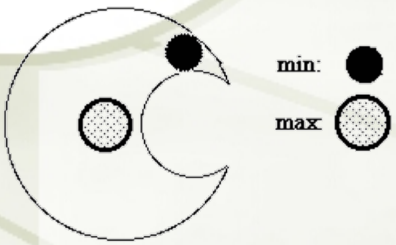


Example 1(b)

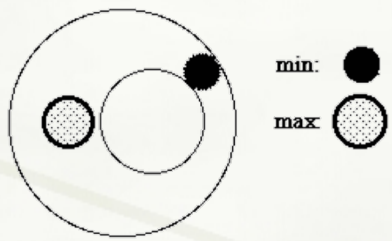


Example 1(c)

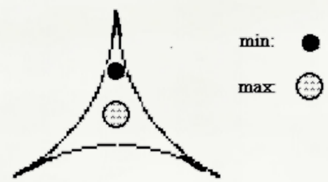
Illustrative examples



Example 1(b)

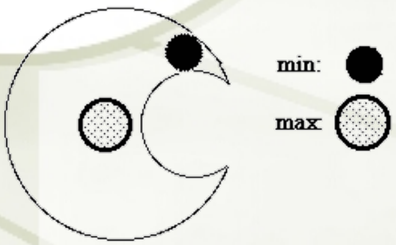


Example 1(c)



Example 8

Illustrative examples



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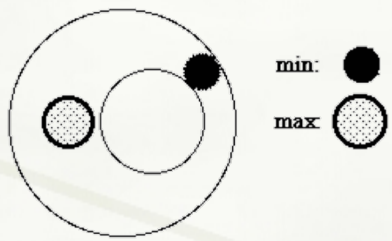

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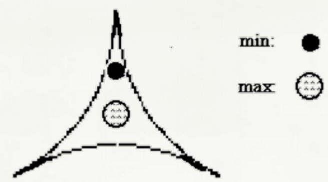
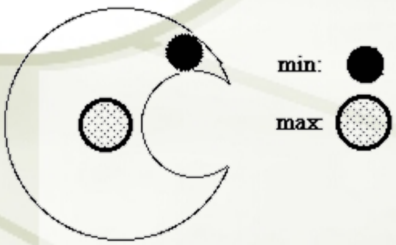


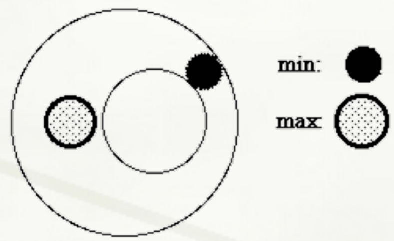
Example 8

Example 4 (a)

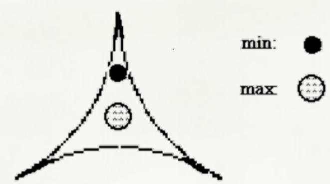
Illustrative examples



Example 1(b)



Example 1(c)



Example 8

Example 4 (a)

Example 11

curious?

Part 2

A new hammer in search of nails.

Sums of eigenvalues

- Suppose that you know about

$$S_k := \sum_{\ell=0}^{k-1} \mu_\ell$$

(say, upper or lower bounds). What else do you know?

Karamata's theorem

Lemma 3.1 (Karamata-Ostrowski) *Let two nondecreasing ordered sequences of real numbers* $\{\mu_j\}$ *and* $\{m_j\}$, $j = 0, \ldots, n-1$, *satisfy*

$$\sum_{j=0}^{k-1} \mu_j \leq \sum_{j=0}^{k-1} m_j \tag{3.7}$$

for each k. *Then for any differentiable convex function* $\Psi(x)$,

$$\sum_{j=0}^{k-1} \Psi(\mu_j) \geq \sum_{j=0}^{k-1} \Psi(m_j) + \Psi'(m_{k-1}) \cdot \sum_{j=0}^{k-1} (\mu_j - m_j).$$

In particular, assuming either that Ψ *is nonincreasing or that* $\sum_{j=0}^{k-1} \mu_j = \sum_{j=0}^{k-1} m_j$,

$$\sum_{j=0}^{k-1} \Psi(\mu_j) \geq \sum_{j=0}^{k-1} \Psi(m_j)$$

Sums of eigenvalues

- With Karamata, inequalities on S_k of the form
$$\sum_{\ell=0}^{k} \mu_\ell \le \sum_{\ell=0}^{k} m_\ell$$
for all k imply further bounds on the trace of the heat kernel, the spectral zeta function, etc.

Spectral averages, geometry, and dimensionality

For Laplacians (DBC):

- Weyl law:

$$\lambda_k \sim 4\pi^2 \left(\frac{k}{C_d|\Omega|}\right)^{2/d}$$

Spectral averages, geometry, and dimensionality

For Laplacians (DBC):

- Weyl law: $$\lambda_k \sim 4\pi^2 \left(\frac{k}{C_d|\Omega|}\right)^{2/d}$$
- Pólya conjectured in the 60's that this is a strict lower bound, and proved it for tiling domains, but this holy grail has still not been proved in full generality.

Spectral averages, geometry, and dimensionality

For Laplacians (DBC):

- Weyl law: $$\lambda_k \sim 4\pi^2 \left(\frac{k}{C_d|\Omega|}\right)^{2/d}$$
- However, averaging helps:

Spectral averages, geometry, and dimensionality

For Laplacians (DBC):

- Weyl law: $$\lambda_k \sim 4\pi^2 \left(\frac{k}{C_d|\Omega|}\right)^{2/d}$$
- Berezin-Li-Yau $$\overline{\lambda_k} := \frac{1}{k}\sum_{\ell=1}^{k} \lambda_\ell \geq \frac{4\pi^2 d}{d+2}\left(\frac{k}{C_d|\Omega|}\right)^{2/d}$$

Spectral averages, geometry, and dimensionality

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- Weyl law: $\lambda_k \sim 4\pi^2(k/C_d|\Omega|)^{2/d}$.
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- Harrell-Hermi* for all k > j

$$\frac{\overline{\lambda_k}}{\overline{\lambda_j}} \leq \frac{4+d}{2+d}\left(\frac{k}{j}\right)^{2/d}$$

JFA '08 cst improved by Harrell-Stubbe, 2011.

Variational bounds on sums

- In 1992 Pawel Kröger found a variational argument for the Neumann counterpart to Berezin-Li-Yau, i.e. a Weyl-sharp upper bounds on sums of the eigenvalues of the Neumann Laplacian.
- BLY: $$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}$$
- Kröger: $$\sum_{j=0}^{k-1} \mu_j \leq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}$$

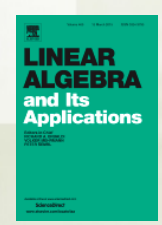
The average hammer

The average hammer

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \le \frac{1}{|\mathfrak{M}_0|}\int_{\mathfrak{M}_0}\frac{Q_M(f_\zeta, f_\zeta)}{\|f_\zeta\|^2}\,d\sigma$$

For any $|\mathfrak{M}_0| = k\frac{A}{C}$

Harrell-Stubbe LAA, 2014



The average hammer

Corollary 1.1. *Under the assumptions of the Theorem, suppose further that* $\|f_\zeta\|^2 = C$ *is independent of* ζ, *and that for all* $\phi \in \mathcal{H}$, $\int_{\mathfrak{M}} |\langle \phi, f_\zeta \rangle|^2 \, d\sigma = A\|\phi\|^2$ *for a fixed* $A > 0$. *Then for any* $\mathfrak{M}_0 \subset \mathfrak{M}$ *such that* $|\mathfrak{M}_0| = k\frac{A}{C}$,

$$\frac{1}{k}\sum_{j=0}^{k-1} \mu_j \le \frac{1}{|\mathfrak{M}_0|}\int_{\mathfrak{M}_0} \frac{Q_M(f_\zeta, f_\zeta)}{\|f_\zeta\|^2} d\sigma. \tag{8}$$

Recent applications of the averaged variational principle:

1. Harrell-Stubbe, LAA 2014: Weyl-type upper bounds on sums of eigenvalues of (discrete) graph Laplacians and related operators.
2. El Soufi-Harrell-Ilias-Stubbe, nearing preprint stage: Semiclassically sharp Neumann boundsfor a large family of 2^{nd} order PDEs.
3. Harrell-Dever, stuff on blackboards: Quantum graphs.
4. Harrell-Stubbe, semiclassically sharp upper bound for Dirtichlet. (Counterpart to Li-Yau.)

Example: Recover Kröger's result

Our theorem says that *IF* $\mathfrak{M}_0$ is sufficiently big that

$$\int_{\mathfrak{M}_0} \langle f_\zeta, f_\zeta \rangle \, d\sigma \geq \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma$$

Then we have an upper bound on a sum involving eigenvalues. For trial functions we take the Fourier exponential functions, and we equate ζ with the dual variable **p**.

Example: Recover Kröger's result

With the Parseval identity,

$$\int_{\mathfrak{M}} |\left\langle e^{i\mathbf{p}\cdot\mathbf{x}}, \psi^{(j)} \right\rangle|^2 = (2\pi)^d \|\psi^{(j)}\|^2 = (2\pi)^d.$$

IF $|\mathfrak{M}_0||\Omega| \geq (2\pi)^d k,$ then

$$(2\pi)^d \sum_{j=0}^{k-1} \mu_j \leq \int_{\mathfrak{M}_0} |\mathbf{p}|^2 |\Omega|$$

Choosing $\mathfrak{M}_0$ as a ball of radius R in p-space,
a simple calculation gives Kröger.

New? *Kröger for Dirichlet*

Using coherent states,

$$f_\zeta = \frac{h(x-y;r)e^{-i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{d/2}}$$

we can get an upper inequality counterpart to Li-Yau.

New? *Kröger for Dirichlet*

If h is normalized in L^2, then for any $\phi \in L^2$, then there is a Parseval-type relation

$$\int \int | \langle f_\zeta, \phi \rangle |^2 dy dp = \|\phi\|^2.$$

Now let's take $0 < r < \mathrm{diam}(\Omega)$ and h the ground-state Dirichlet eigenfunction of the Ball of radius r. *If* $\mathbf{y} \in \Omega : \mathrm{dist}(\mathbf{y}, \partial\Omega) \geq r$, then the energy form in our inequality is

$$\langle \nabla f_\zeta, \nabla f_\zeta \rangle = |\mathbf{y}|^2 + \frac{\mathcal{J}}{r^2}.$$

If we average this over a subset of (y,p) of the right size*, we get an upper bound for the sum of the first k Dirichlet eigenvalues.

* $$|\mathfrak{M}_0| = (2\pi)^d k.$$

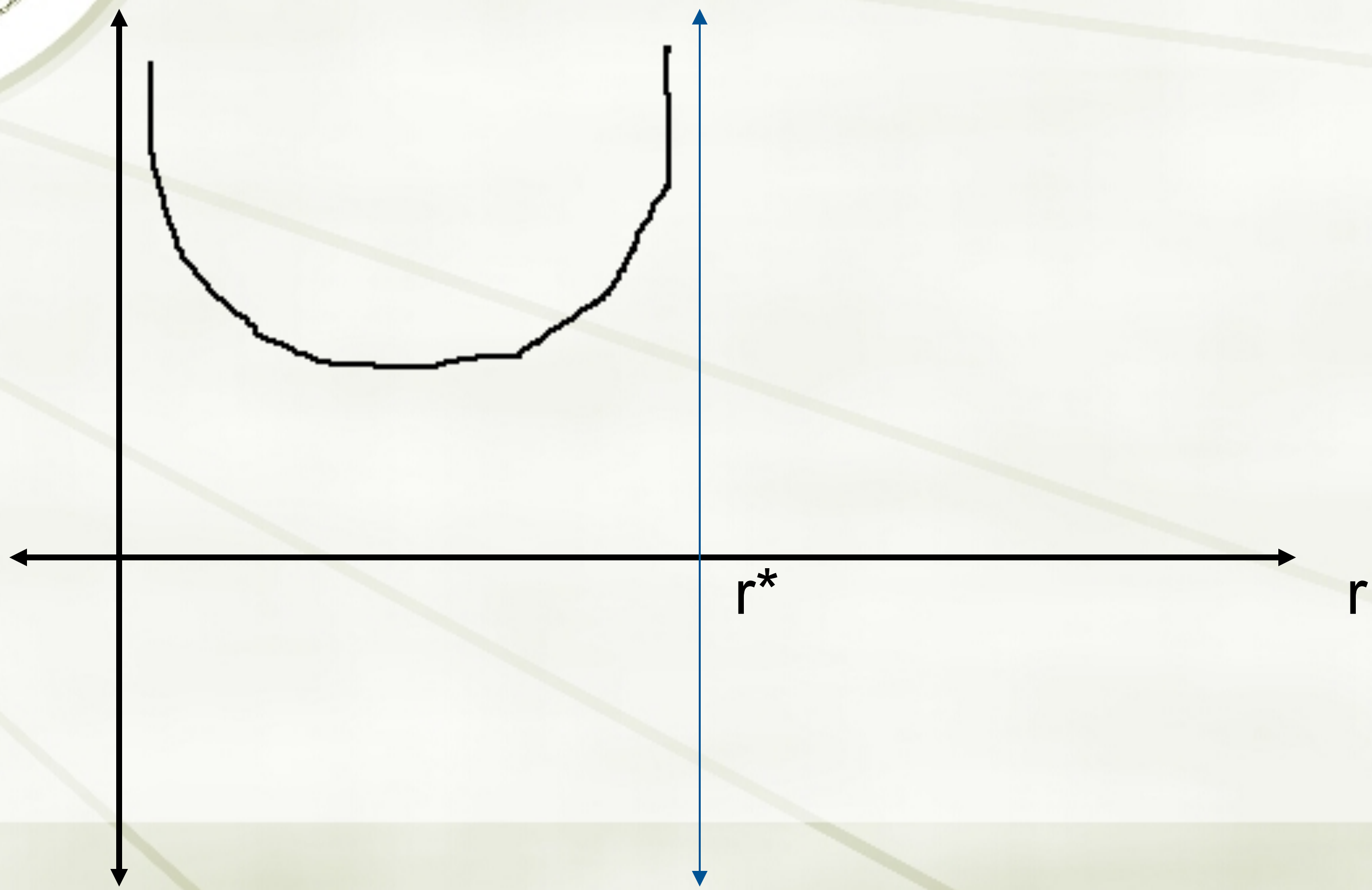
New? Kröger for Dirichlet

For each r>0 such that the retract Ω_r has positive volume,

$$\overline{\lambda_k} := \frac{1}{k}\sum_{\ell}^{k} \lambda_\ell \le \frac{d+2}{d}(4\pi)^2 \left(\frac{k}{|\Omega_r||B_1,d|}\right)^{\frac{2}{d}} + \frac{\mathcal{J}}{r^2}.$$

We let **y** range over the retracted set Ω_r and (in the usual way for Li-Yau) let **p** range over a ball of the required size.

$$\frac{d+2}{d}(4\pi)^2\left(\frac{k}{|\Omega_r||B_1,d|}\right)^{\frac{2}{d}}+\frac{\mathcal{J}}{r^2}.$$



Spectral dimension

- Notice that you can unambiguously determine the volume and dimension from these inequalities.
- We can refer to the optimal exponent in a BLY or Kröger-type bound as defining the *spectral dimension* d by interpreting the power of k in this pair of inequalities as $1 + 2/d$.

Spectral dimension

- We can use the optimal exponent in a BLY or Kröger-type bound to define the *spectral dimension*.
- Dimension in the ordinary sense is a *measure of complexity*.

Spectral dimension

- We can use the optimal exponent in a BLY or Kröger-type bound to define the *spectral dimension*.
- Dimension in the ordinary sense is a *measure of complexity*.
- How closely can we tie the spectral dimension to a geometric dimension?

Combinatorial graphs

- A graph connects n vertices with edges as specified by an adjacency matrix A, with $a_{ij} = 1$ when i and j are connected, otherwise 0. The graph is not a priori living in Euclidean space. *But it might be! Clearly it could at worst be embedded in $\mathbb{Q}^{n-1}$, but what's the minimal dimension?*

Combinatorial graphs

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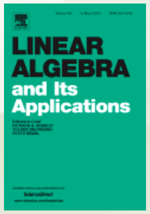
- *Note that considering a graph as a subgraph of a regular lattice graph is quite different from just drawing it in R^n.*

More vs less efficient embeddings of a 1D graph.

Combinatorial graphs

- We use the graph Laplacian to get conditions for embeddability.

Harrell-Stubbe LAA, 2014



Connecting the spectrum of a graph and its embedding dimension

$$\overline{\lambda_k} \leq 2m\kappa \left(1 - \mathrm{sinc}(\kappa^{1/d}\pi)\right),$$

where $\kappa := \frac{k}{n}$, and $sinc(x) := \frac{sin(x)}{x}$.

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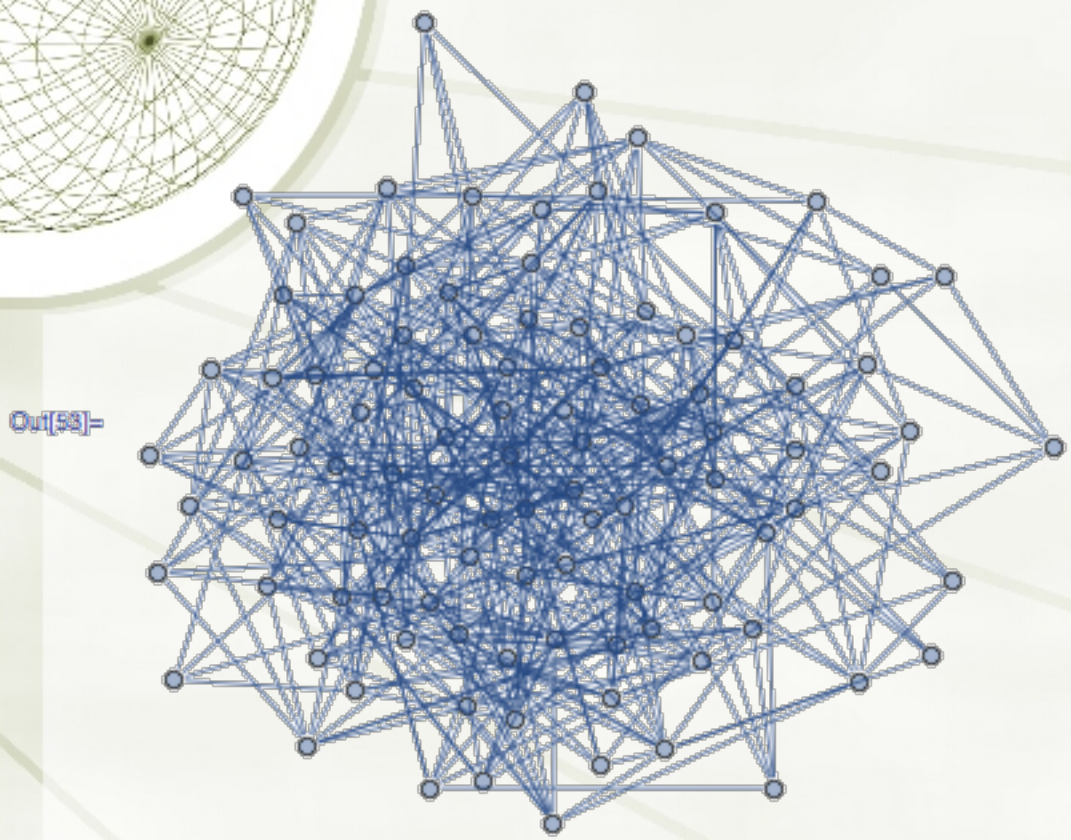
With Taylor,

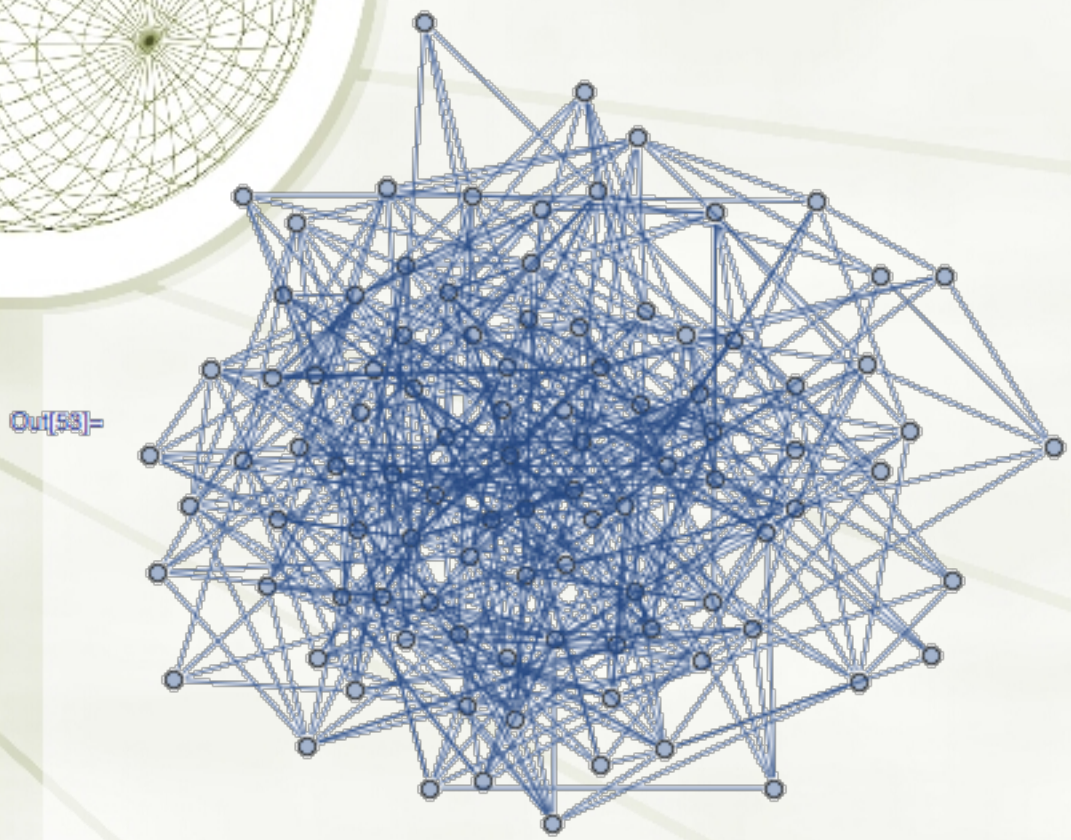
$$\overline{\lambda_k} \leq \frac{\pi^2 m}{3}\kappa^{\frac{2}{d}} - \dots$$

Dimension and complexity



This is a randomly generated "graph" showing 520 connections among 100 items. What is the intrinsic dimensionality?

Dimension and complexity



You might not see it visually, but the spectrum says that this is 3D!

Another interesting question:

Can you distinguish dimensions on different scales?

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Bounds on sums for quantum graphs

Let us assume that our quantum graph consists of a finite number of straight lines, which can be isometrically embedded in d-dimensional Euclidean space. We'll define the Hamiltonian as the Friedrichs extension of the quadratic form

$$Q(\varphi, \varphi) = \sum_{\mathcal{E} \subset \Gamma} \int_{\mathcal{E}} \left(|\varphi'|^2 + V(x)|\varphi|^2 \right) dx$$

on functions $\varphi \in H^1(\Gamma)$, interpreted as the orthogonal sum of H^1 on the edges, with Lebesgue measure.

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For today we'll set V=0, and avoid the temptation to introduce other complications.
Well, other than some general remarks.

Bounds on sums for quantum graphs

To get the machine running, we’d like a set of trial functions which have a nice relation to the operator and a completeness relation, so the Fourier exponentials again come to mind.

An adapted Fourier transform

If $f(x) \in L^2(\Gamma)$, we can define a Fourier transform adapted to the graph by

$$\hat{f}(\mathbf{p}) := \frac{1}{\sqrt{2\pi}} \int_\Gamma f(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{p}}$$

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Inverse transform:

$$\check{g}(\mathbf{x}) = \mathfrak{F}^{-1}[g]\,\mathbf{x} := \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \frac{1}{(2\pi n)^{d-1}} \int g(\mathbf{p}) e^{i\mathbf{x}\cdot\mathbf{p}} d^d p$$

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Parseval relation:

$$\langle f, g \rangle = \left\langle \hat{f}, \hat{g} \right\rangle_* = \lim_{n\to\infty} \frac{1}{(2\pi n)^{d-1}} \int_{-\pi n}^{\pi n} \cdots \int_{-\pi n}^{\pi n} \hat{f}\overline{\hat{g}} d^d p$$

Bounds on sums for quantum graphs

So, what happens when you use $f_\zeta = \frac{1}{\sqrt{2\pi}} e^{-i\mathbf{p}\cdot\mathbf{x}}$ as a test function in the AVP?

$$Q(f_\zeta, f_\zeta) = \sum_{\mathcal{E}\subset\Gamma} \int_{\mathcal{E}} |p_{\mathcal{E}}|^2 =: q_\Gamma(\mathbf{p}),$$

Bounds on sums for quantum graphs

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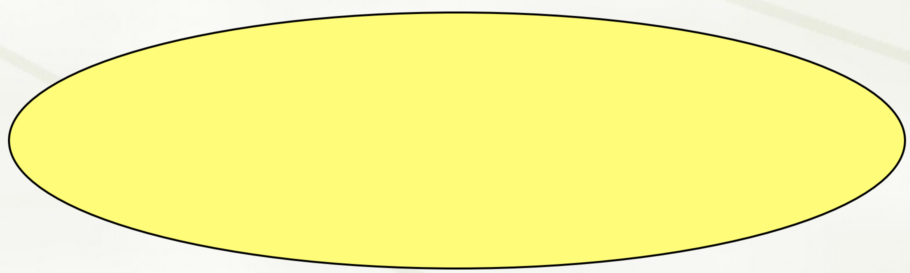
$$Q(f_\zeta, f_\zeta) = \sum_{\mathcal{E}\subset\Gamma} \int_{\mathcal{E}} |p_{\mathcal{E}}|^2 =: q_\Gamma(\mathbf{p}),$$

which defines a certain *phase-space ellipsoid* via $q_\Gamma^{-1}(1)$.



Bounds on sums for quantum graphs

The game is to minimize the integral of $q(\mathbf{p})$, which we think of as the semi-classical energy, over a region in phase space, subject to the region being "sufficiently large" according to the A.V.P.

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Using the bathtub principle, this is done by minimizing the mean energy of a phase-space region defined by

$$\{\mathbf{p} : q(\mathbf{p}) \leq \Lambda\},$$

Bounds on sums for quantum graphs

Now, if q is homogeneous function of degree h, with a scaling argument, we find that

$$\int_{q(\mathbf{p})\leq\Lambda} q(\mathbf{p})d^d p = \frac{d}{d+h}(V(1))^{-d/h}\left(V(\Lambda)\right)^{1+\frac{h}{d}},$$

where $V(\Lambda)$ designates the volume of the phase-space region with maximum energy Λ. (Here $h=2$, but other powers are sometimes of interest.) By the theorem, we need to set $V(\Lambda) = \frac{k}{|\Gamma|}$.

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where $V(\Lambda)$ designates the volume of the phase-space region with maximum energy Λ. (Here $h=2$, but other powers are sometimes of interest.) By the theorem, we need to set $V(\Lambda)=\frac{k}{|\Gamma|}$.

Conclusion. For quantum graphs,

$$\sum_{\ell=0}^{k}\mu_k \leq \frac{d}{d+2}(V(1))^{-d/2}\left(\frac{k}{|\Gamma|}\right)^{1+\frac{2}{d}}$$

Bounds on individual eigenvalues

In the derivation of the AVP we threw something out:

$$\mu_k \left(\cancel{\int_{\mathfrak{M}_0} \langle f_\zeta, f_\zeta \rangle d\sigma - \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma} \right)$$

$$\le$$

$$\int_{\mathfrak{M}_0} \langle H f_\zeta, f_\zeta \rangle d\sigma - \sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma,$$

But you *don't have to throw that part out*, and using the earlier result, you can circle back and get bounds on individual eigenvalues. With some work…

Bounds on sums for quantum graphs

$$m_k(1 - \sqrt{1 - S_k}) \le \mu_{k+1} \le m_k(1 + \sqrt{1 - S_k})$$

$$S_k = \frac{\frac{d+2}{d} \frac{1}{k} \sum_{i=1}^{k} \mu_i}{m_k}.$$

This result is in the form that applies to the case of Euclidean domains, where m_k is the Weyl expression, but a similar result works for all of our applications, including quantum graphs. (Harrell-Stubbe, unpublished)

PDEs on Riemannian manifolds, and phase-space bounds

Variational bounds on sums

- Berezin-Li-Yau and Kröger have been extended to manifolds of various kinds. In particular, Strichartz understood that Kröger's argument works on subdomains of homogeneous spaces other than R^d.

The mother of all upper bounds on sums for PDEs

We (El Soufi, Harrell, Ilias, Stubbe) recently used the A.V.P. to get upper bounds for sums of eigenvalues of corresponding to quadratic forms.

$$\mathcal{E}(\varphi) := \frac{\int_\Omega (|\nabla\varphi(\mathbf{x})|^2 + V(\mathbf{x})|\varphi(\mathbf{x})|^2) w(\mathbf{x}) e^{-2\rho(\mathbf{x})} dv_g}{\int_\Omega |\varphi(\mathbf{x})|^2 e^{-2\rho(\mathbf{x})} dv_g},$$

where Ω is a domain in a general Riemannian manifold.

Some Kröger-type results for general Riemannian manifolds

- Maybe I'd better just summarize some of the highlights.
  1. There is an adapted F.T. for any Riemannian manifold, but the Parseval relation becomes an inequality involving the so-called Riemannian constant.

An adapted Fourier transform

Let $F : (M, g) \to \mathbb{R}^N$, be an isometric embedding (whose existence for sufficiently large N is guaranteed by Nash's embedding Theorem). To any function $u \in L^2(\Omega)$, we associate the function $\hat{u}_F : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\hat{u}_F(\mathbf{p}) = \int_\Omega u(\mathbf{x}) e^{i\mathbf{p}\cdot F(\mathbf{x})} dv_g$$

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(see [1, Theorem 2.1] [10, Theorem 7.1.26], [16, Corollary 5.2]), that there exists a constant $C_{F(\Omega)}$ such that, $\forall u \in L^2(\Omega)$ and $\forall R > 0$,

$$\int_{B_R} |\hat{u}_F(\mathbf{p})|^2 d^N p \leq C_{F(\Omega)} R^{N-\nu} \|u\|^2 \tag{17}$$

where B_R is the Euclidean ball of radius R in $\mathbb{R}^N$ centered at the origin and $\|u\|^2 = \int_\Omega u^2 dv_g$.

We define the Riemannian constant H_Ω by

$$H_\Omega = \inf_{N \geq \nu} \inf_{F \in I(M, \mathbb{R}^N)} \left(\frac{\nu + 2}{N + 2} \right)^{\frac{\nu}{2}} \frac{1}{\omega_N} C_{F(\Omega)} \tag{18}$$

Theorem 3.1. *Let (M,g) be a Riemannian manifold of dimension $\nu \geq 2$. Let $\mu_l = \mu_l(\Omega, g, \rho, w, V)$, $l \in \mathbb{N}$, be the eigenvalues defined by (2) and (3) on a bounded open set $\Omega \subset M$, where w, ρ, and V satisfy the assumptions stated above. Then*

(1) For all $z \in \mathbb{R}$,

$$\sum_{j\geq 0}(z-\mu_j)_+ \geq \frac{2\ |\Omega|_g}{(\nu+2)H_\Omega}\left(\fint_\Omega w\,dv_g\right)^{-\frac{\nu}{2}}\left(z-\fint_\Omega \widetilde{V} w\,dv_g\right)_+^{1+\frac{\nu}{2}} \tag{20}$$

where $\widetilde{V} = V + |\nabla^g \rho|^2$.

(2) For all $k \in \mathbb{N}$,

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \leq \frac{\nu}{\nu+2}\left(\frac{H_\Omega}{|\Omega|_g}k\right)^{\frac{2}{\nu}}\fint_\Omega w\,dv_g + \fint_\Omega \widetilde{V} w\,dv_g. \tag{21}$$

(3) For all $t > 0$,

$$\sum_{j\geq 0} e^{-t(\mu_j - \fint_\Omega \widetilde{V} w\,dv_g)} \geq \left(\frac{\pi}{t}\right)^{\frac{\nu}{2}}\ \frac{|\Omega|_g}{\omega_\nu H_\Omega}\left(\fint_\Omega w\,dv_g\right)^{-\frac{\nu}{2}}. \tag{22}$$

If the manifold is conformal to a homogeneous spaces, more precise bounds are obtained

A homogeneous space is a manifold M with a continuous symmetry group of isomorphisms $M \to M$.

Canonical examples: R^d, S^d, H^d.

Some Kröger-type results for general Riemannian manifolds

- More highlights.
  1. If the space is homogeneous, there is an exact Parseval identity, so we get something very much like the Kröger result. (cf. Strichartz)
  2. But even better, we get just as sharp results for the “mother of all” expression, i.e., with weights and a generic conformal transformation.

Corollary 3.4. *Let Ω be a bounded domain of $\mathbb{R}^\nu$ and let $g = \alpha^2 g_E$ be a Riemannian metric that is conformal to the Euclidean metric g_E. The Neumann eigenvalues $\{\mu_l\}$ of the Laplacian Δ_g in Ω satisfy the following estimates in which $|\Omega|$ denotes the Euclidean volume of Ω :*

(1) For all $z \in \mathbb{R}$,

$$\sum_{j\geq 0} (z-\mu_j)_+ \geq \frac{2\omega_\nu|\Omega|}{(\nu+2)(2\pi)^\nu} \left(\fint_\Omega \alpha^{-2}\, d^\nu x \right)^{\frac{\nu}{2}} \left(z - \frac{\nu^2}{4} \fint_\Omega |\nabla\alpha|^2 \alpha^{-4}\, d^\nu x \right)_+^{1+\frac{\nu}{2}} \tag{44}$$

(2) For all $k \in \mathbb{N}$,

$$\frac{1}{k}\sum_{j=0}^{k-1} \mu_j \leq \frac{\nu}{\nu+2} 4\pi^2 \left(\frac{k}{\omega_\nu|\Omega|} \right)^{\frac{2}{\nu}} \fint_\Omega \alpha^{-2}\, d^\nu x + \frac{\nu^2}{4} \fint_\Omega |\nabla\alpha|^2 \alpha^{-4}\, d^\nu x. \tag{45}$$

(3) For all $k \in \mathbb{N}$,

$$\mu_k \left(1 - \frac{\nu^2}{4} \frac{\fint_\Omega |\nabla\alpha|^2\alpha^{-4}\, d^\nu x}{\mu_k} \right)_+^{1+\frac{2}{\nu}} \leq 4 \left(\frac{\nu+2}{2} \right)^{\frac{2}{\nu}} \pi^2 \left(\frac{k}{\omega_\nu|\Omega|} \right)^{\frac{2}{\nu}} \fint_\Omega \alpha^{-2}\, d^\nu x. \tag{46}$$

In particular,

$$\mu_k \leq \max\left\{ \frac{\nu^2}{2} \fint_\Omega |\nabla\alpha|^2\alpha^{-4}\, d^\nu x \ ; \ 8\,(\nu+2)^{\frac{2}{\nu}}\, \pi^2 \left(\frac{k}{\omega_\nu|\Omega|} \right)^{\frac{2}{\nu}} \fint_\Omega \alpha^{-2}\, d^\nu x \right\}. \tag{47}$$

For example, a domain of the hyperbolic space $\mathbf{H}^\nu$ can be identified with a domain of the Euclidean unit ball endowed with the metric $g = \left(\frac{2}{1-|x|^2}\right)^2 g_E$. Corollary 3.4 gives for such a domain, with $\alpha = \frac{2}{1-|x|^2}$, $\fint_\Omega \alpha^{-2}\, d^\nu x \leq \frac{1}{4}$ and $\fint_\Omega |\nabla\alpha|^2\alpha^{-4}\, d^\nu x = \fint_\Omega |x|^2\, d^\nu x$,

Some Kröger-type results for general Riemannian manifolds

- More highlights.

  3. When you include a potential, the region of phase space that makes physical sense is $p^2 + V(x) \leq \Lambda$. We obtain semiclassically sharp results in this case with *coherent states*.

Coherent states

For domains conformal to Euclidean sets, we take

$$f_\varsigma(\mathbf{x}) := \frac{1}{(2\pi)^{\nu/2}} e^{i\mathbf{p}\cdot(\mathbf{x})+\rho(\mathbf{x})} h(\mathbf{x}-\mathbf{y}).$$

and reason as follows

Some definitions

- The *effective potential* $\widetilde{V}(\mathbf{x}) := V(\mathbf{x}) + |\nabla\rho|^2(\mathbf{x})$;
- The Euclidean *phase-space volume* for energy Λ,

$$\Phi(\Lambda) := |(\mathbf{x},\mathbf{p}) : |\mathbf{p}|^2 + \widetilde{V}(\mathbf{x}) \le \Lambda| = \omega_\nu \int \left(\Lambda - \widetilde{V}(\mathbf{x})\right)_-^{\frac{\nu}{2}} d^\nu x,$$

- The *weighted phase-space volume* and *weighted total energy*

$$\Phi_w(\Lambda) = \omega_\nu \int \left(\Lambda - \widetilde{V}(\mathbf{x})\right)_-^{\frac{\nu}{2}} w(\mathbf{x}) d^\nu x$$

$$\begin{aligned} E_w(\Lambda) &:= \int_{\{(\mathbf{x},\mathbf{p}):|\mathbf{p}|^2+\widetilde{V}(\mathbf{x})\le\Lambda\}} \left(|\mathbf{p}|^2 + \widetilde{V}(\mathbf{x})\right) w(\mathbf{x}) d^\nu x \\ &= \frac{\nu}{\nu+2}\omega_\nu \int_\Omega \left(\Lambda - \widetilde{V}(\mathbf{x})\right)_-^{1+\frac{\nu}{2}} w(\mathbf{x}) d^\nu x. \end{aligned}$$

Some definitions

- The L^2-normalized ground-state Dirichlet eigenfunction for the ball of geodesic radius r in M will be denoted h_r and $\mathcal{K}(h_r) := \int_{B_r} |\nabla h_r(\mathbf{x})|^2 d^\nu x$. I.e., in this section where $M = \mathbb{R}^\nu$, h is a scaled Bessel function and

$$\mathcal{K}(h_r) = \frac{j^2_{\frac{\nu}{2}-1,1}}{r^2}.$$

- $L(\Lambda)$ will denote the maximal Lipschitz constant of $\widetilde{V}(\mathbf{x})$ on the region $\Omega \cap \{\mathbf{x} : \widetilde{V}(\mathbf{x}) \leq \Lambda\}$.

Definition 4.2. *The* Euclidean phase-space volume *for energy* Λ *is defined as*

$$\Phi_1(\Lambda) := \frac{1}{(2\pi)^\nu} |(\mathbf{x},\mathbf{p}) : |\mathbf{p}|^2 + \widetilde{V}(\mathbf{x}) \le \Lambda| = \frac{\omega_\nu}{(2\pi)^\nu} \int_\Omega \left(\Lambda - \widetilde{V}(\mathbf{x})\right)_+^{\frac{\nu}{2}} d^\nu x,$$

according to a standard calculation to be found, for example, in [13]*. Here* ω_ν *is the volume of the unit ball in dimension* ν *and* $(x)_+ := max(x, 0)$*. If the weight in* (2) *is not constant, we make use of a* weighted phase-space volume,

$$\Phi_w(\Lambda) = \frac{\omega_\nu}{(2\pi)^\nu} \int_\Omega \left(\Lambda - \widetilde{V}(\mathbf{x})\right)_+^{\frac{\nu}{2}} w(\mathbf{x}) d^\nu x.$$

The total energy associated with this quantity is correspondingly

$$\begin{aligned} E_w(\Lambda) &:= \frac{1}{(2\pi)^\nu} \int_{\{(\mathbf{x},\mathbf{p}):\mathbf{x}\in\Omega, |\mathbf{p}|^2+\widetilde{V}(\mathbf{x})\le\Lambda\}} \left(|\mathbf{p}|^2 + \widetilde{V}(\mathbf{x})\right) w(\mathbf{x}) d^\nu x d^\nu p \\ &= \frac{\nu}{\nu+2} \frac{\omega_\nu}{(2\pi)^\nu} \int_\Omega \left(\Lambda - \widetilde{V}(\mathbf{x})\right)_+^{1+\frac{\nu}{2}} w(\mathbf{x}) d^\nu x. \end{aligned} \tag{51}$$

Theorem 4.1. *Let $\mu_0 \leq \mu_1 \leq \dots$ be the variationally defined Neumann eigenvalues for the quadratic form (2) on an open set $\Omega \in \mathbb{R}^\nu$, where w, ρ, and V satisfy the assumptions stated above, and define $\Lambda(k)$ as the minimal value of Λ for which $\Phi_1(\Lambda) \geq (2\pi)^\nu k$. Then*

$$\sum_{j=0}^{k-1} \mu_j \leq E_w(\Lambda(k)) + 3\left(2 j_{\nu-1,1}^2 \mathrm{Lip}(\Lambda(k))\right)^{\frac{1}{3}} \Phi_w\left(\Lambda(k) + (2 j_{\nu-1,1}^2 \mathrm{Lip}(\Lambda(k)))^{\frac{1}{3}}\right). \tag{53}$$

THE END