Universal estimates of eigenvalues of elliptic operators

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12 September 2007

Eigenvalues of Laplace and Schrödinger operators

- Laplace operator. Eigenvalues are squares of vibration frequencies

$-\Delta u_j = \lambda_j u_j$, Dirichlet BC

$\lambda_j > 0$

$$\|u_j\|_2 = 1$$

- $0 \leq \lambda_j \to \infty$, but…

- $0 \leq \lambda_j \rightarrow \infty$, but not just any increasing sequence of positive numbers can be the spectrum of a Laplace operator! It is very far from the truth.

- Not just any sequence of positive numbers can be the spectrum of a Laplace operator!
- Likewise for Schrödinger operators,

$$H = -\Delta + V,$$

$$H u_j = \lambda_j u_j$$

- Not just any sequence of positive numbers can be the spectrum of a Laplace operator!
- Likewise for Schrödinger operators,

$$H = -\Delta + V.$$

- We shall assume discrete spectrum.

"Universal" constraints on eigenvalues

- H. Weyl, 1910, Laplace operator (V=0),

$$\lambda_n \sim (4\pi) \left(\frac{\Gamma(1 + \frac{d}{2}) n}{|\Omega|} \right)^{\frac{2}{d}}$$

“Universal” constraints on eigenvalues

- H. Weyl, 1910, Laplace operator,

$$\lambda_n \sim (4\pi)\left(\frac{\Gamma(1+\frac{d}{2})n}{|\Omega|}\right)^{\frac{2}{d}}$$

- Pólya conjecture, 1961: This is a lower bound.

"Universal" constraints on eigenvalues

- E. Lieb and W. Thirring, 1977; Berezin, 1972, P. Li and S.T. Yau, 1983. Look at sums, moments

$$\sum_{j=1}^{k} \lambda_j \qquad \sum_{j=1}^{k} \lambda_j^{p}$$

as k $\rightarrow\infty$.

"Universal" constraints on eigenvalues

- Berrezin-Li-Yau:

$$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}$$

"Universal" constraints on eigenvalues

Low-lying eigenvalues:

- W. Kuhn, F. Reiche, W. Thomas, 1925, sum rules for energy eigenvalues of Schrödinger operators;
- Heisenberg, 1925, connected TRK sum rules to commutator relations.

"Universal" constraints on eigenvalues

- L. Payne, G. Pólya, H. Weinberger, 1956: gaps between eigenvalues of Laplacian controlled by sums of lower eigenvalues:

- $\lambda_{n+1} - \lambda_n \leq (4/dn) \Sigma_{k\leq n} \lambda_k$

- For example, for the *fundamental gap*,

  $G := \lambda_2 - \lambda_1 \leq (4/d)\lambda_1$, i.e.,

  $\lambda_2/\lambda_1 \leq 1 + 4/d$.

"Universal" constraints on eigenvalues

- Hile-Protter, 1980, stronger but more complicated analogue of PPW.
- Ashbaugh-Benguria 1991, isoperimetric conjecture of PPW proved: λ_2/λ_1 maximized by ball. (Not far off of PPW 1 + 4/d .)
- H. Yang 1991-5, unpublished, complicated formulae like PPW, respecting Weyl asymptotics.

“Universal” constraints on eigenvalues

- A philosophy, or if you will, a prejudice:

  A universal bound on eigenvalues is of purely algebraic origin.

- Nonetheless, the original proofs of the inequalities mentioned above were heavily analytic, making essential use of min-max.

“Universal” constraints on eigenvalues

- Commutators: $[H, G] := HG - GH$

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  - Calculate in sense of operators, e.g.

    [d/dx , x] = 1,

    for given any differentiable function f,

$$\left[\frac{d}{dx}, x\right] f = \frac{d}{dx}(xf) - x\frac{d}{dx}f = f = 1f$$

"Universal" constraints on eigenvalues

- Commutators: [H, G] := HG - GH
  - Calculate in sense of operators, e.g.,

    [d/dx , x] = 1,
  - The elementary gap formula:,

$$(\lambda_j - \lambda_k) \langle u_j, G\, u_k \rangle = \langle H\, u_j, G\, u_k \rangle - \langle u_j, G\, H\, u_k \rangle$$

$$= \langle u_j, [H,G]\, u_k \rangle$$

Commutators and gaps

Elementary gap formula:

$$\langle u_j, [H,G]u_k\rangle = (\lambda_j - \lambda_k)\langle u_j, Gu_k\rangle . \tag{1.2}$$

Since $[H,G]u_k = (H - \lambda_k)Gu_k,$

$$\|[H,G]u_k\|^2 = \left\langle Gu_k, (H-\lambda_k)^2 Gu_k \right\rangle , \tag{1.3}$$

and more generally

$$\langle [H,G]u_j, [H,G]u_k\rangle = \langle Gu_j, (H-\lambda_j)(H-\lambda_k)Gu_k\rangle . \tag{1.4}$$

Second commutator formula:

$$\langle u_j \mid [G,[H,G]]\, u_k\rangle = \langle Gu_j \mid (2H - \lambda_j - \lambda_k)\, Gu_k\rangle . \tag{1.5}$$

In particular,

$$\langle u_j \mid [G,[H,G]]\, u_j\rangle = 2\,\langle Gu_j \mid (H - \lambda_j)\, Gu_j\rangle . \tag{1.6}$$

Commutators and gaps

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Traces

Trace has convenient properties:

- Linear
- $Tr(BA) = Tr(AB)$
- $Tr(f(H)) = \sum f(\lambda_k)$

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To relate the commutators we multiply expressions by a spectral projector of H and take traces.

A general trace formula

Assuming that H, G are self-adjoint and that the spectrum of H is purely discrete,

$$(z-\lambda_j)\,\langle [G,[H,G]]\,u_j,u_j\rangle - 2\|\,[H,G]\,u_j\,\|^2$$

$$= 2\sum_k (z-\lambda_k)(\lambda_k-\lambda_j)|\,\langle G u_j, u_k\rangle\,|^2$$

Canonical commutation

- Suppose now that H is a Schrödinger operator of standard type, $H = - \nabla^2 + V(x)$, on a Euclidean domain, and that G is a Euclidean coordinate x_k. Then $[H,G] = -2\, \partial / \partial x_k$, and the second commutator $[G, [H, G]] = 2$.

Canonical commutation

- Suppose now that H is a Schrödinger operator of standard type, H = - Ñ2 + V(x), on a Euclidean domain, and that G is a Euclidean coordinate xk. Then [H,G] = -2∂ / ∂ xk, and the second commutator [G, [H, G]] = 2 .
- *Physical interpretation*: Up to scalar factors, [H,G] is a momentum, and [G, [H, G]] = 2 is a form of the Heisenberg commutation relation.

Universal Bounds using Commutators

- A “sum rule” identity (Harrell-Stubbe, 1997):

$$1 = \frac{4}{d} \sum_{k:\lambda_k \neq \lambda_j} \frac{|\langle u_k, \mathbf{p} u_j \rangle|^2}{\lambda_k - \lambda_j}$$

Here, H is *any* Schrödinger operator, **p** is the gradient
(times -i if you are a physicist and you use “atomic units”)

Universal Bounds using Commutators

$$1 = \frac{4}{d} \sum_{k:\lambda_k \neq \lambda_j} \frac{|\langle u_k, \mathbf{p} u_j \rangle|^2}{\lambda_k - \lambda_j}$$

Idea: Multiply by a function of λ_j and sum on j. A good choice is the function $(z - \lambda_j)^2$, which leads after some work to:

$$\sum_{j:\lambda_j < z} (z - \lambda_j)^2 \leq \frac{d}{4} \sum_{j:\lambda_j < z} (z - \lambda_j)(\lambda_j - \langle u_j, V u_j \rangle)$$

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→ 0 for $-\Delta$ or $V \geq 0$

Higher power ≤ lower

Riesz means and how to get information from them

$$R_\sigma(z) := \sum_\ell (z - \lambda_\ell)_+^\sigma$$

Riesz means and how to get spectral information from them

These ideas will be illustrated for the Laplacian on a Euclidean domain (joint work with L. Hermi).

Universal bounds of the form λ_k / λ_1

Bounds of this form follow easily from the earlier bounds on λ_{k+1}, but with bad constants.

Universal bounds of the form λ_k / λ_1

Some previous work:

Universal bounds of the form λ_k / λ_1

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Ashbaugh-Benguria, 1994:

$$\frac{\lambda_{2^m}}{\lambda_1} \leq \left(\frac{j^2_{d/2,1}}{j^2_{d/2-1,1}} \right)^m$$

Not of Weyl type. One hopes for $\lambda_k \sim C\, k^{2/d}$.

Hermi, TAMS to appear:

$$\frac{\lambda_{k+1}}{\lambda_1} \le 1 + \left(1 + \frac{d}{2}\right)^{2/d} H_d^{2/d} \, k^{2/d},$$

and

$$\frac{\overline{\lambda}_k}{\lambda_1} \le 1 + \frac{H_d^{2/d}}{1 + \frac{2}{d}} \, k^{2/d},$$

H_d is a constant involving zeroes of Bessel functions.

Cheng-Yang, Math. Ann., 2007:

$$\frac{\lambda_{k+1}}{\lambda_1} \leq C_0(d,k)k^{\frac{2}{d}}$$

where in its simplest form, $C_0 = (1 + 4/d)$.

When d=2, the CY bound is more than 4 times the Weyl asymptotics,

$$\frac{4\left(\Gamma(1+\frac{d}{2})\right)^{\frac{4}{d}}}{j^2_{\frac{d}{2}-1,1}} k^{\frac{2}{d}}$$

Ratios of Averages

$$\overline{\lambda_k} := \frac{1}{k}\sum_{\ell\leq k}\lambda_\ell$$

$$\overline{\lambda_j^2} := \frac{1}{j}\sum_{\ell\leq j}\lambda_\ell^2$$

Ratios of Averages

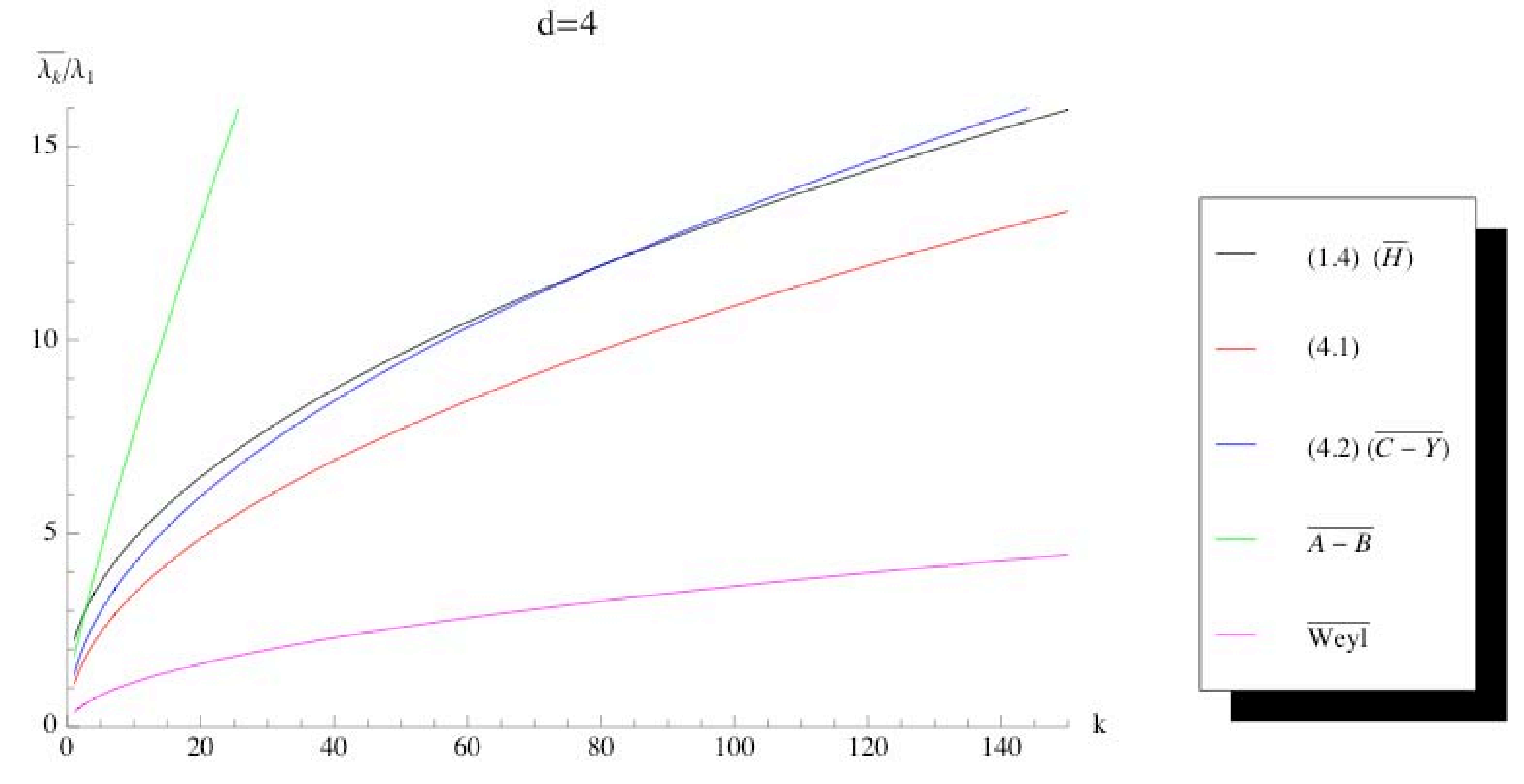
Corollary 3.1 *For $k \geq j^{\frac{1+\frac{d}{2}}{1+\frac{d}{4}}}$, the means of the eigenvalues of the Dirichlet Laplacian satisfy a universal Weyl-type bound,*

$$\overline{\lambda_k}/\overline{\lambda_j} \leq 2\left(\frac{1+\frac{d}{4}}{1+\frac{d}{2}}\right)^{1+\frac{2}{d}}\left(\frac{k}{j}\right)^{\frac{2}{d}}. \qquad (3.4)$$

Ratios of Averages to λ_1

Corollary 3.2 *For* $k \geq \frac{(d+1)\,(1+\frac{d}{2})}{1+\frac{d}{4}}$,

$$\overline{\lambda_k}/\lambda_1 \leq \frac{d+5}{2^{\frac{2}{d}}} \left(\frac{(d+4)}{(d+1)(d+2)} \right)^{1+\frac{2}{d}} k^{\frac{2}{d}}. \qquad (3.5)$$


d=4
$\overline{\lambda_k}/\lambda_1$
15
10
5
0
0
20
40
60
80
100
120
140
k
(1.4) ($\overline{H}$)
(4.1)
(4.2) ($\overline{C-Y}$)
$\overline{A-B}$
$\overline{\text{Weyl}}$

Theorem 2.1 *For $0 < \sigma \leq 2$ and $z \geq \lambda_1$,*

$$R_{\sigma-1}(z) \geq \left(1 + \frac{d}{4}\right) \frac{1}{z} R_\sigma(z); \tag{2.2}$$

$$R'_\sigma(z) \geq \left(1 + \frac{d}{4}\right) \frac{\sigma}{z} R_\sigma(z); \tag{2.3}$$

and consequently

$$\frac{R_\sigma(z)}{z^{\sigma + \frac{d\sigma}{4}}}$$

is a nondecreasing function of z.

For $2 \leq \sigma < \infty$ and $z \geq \lambda_1$,

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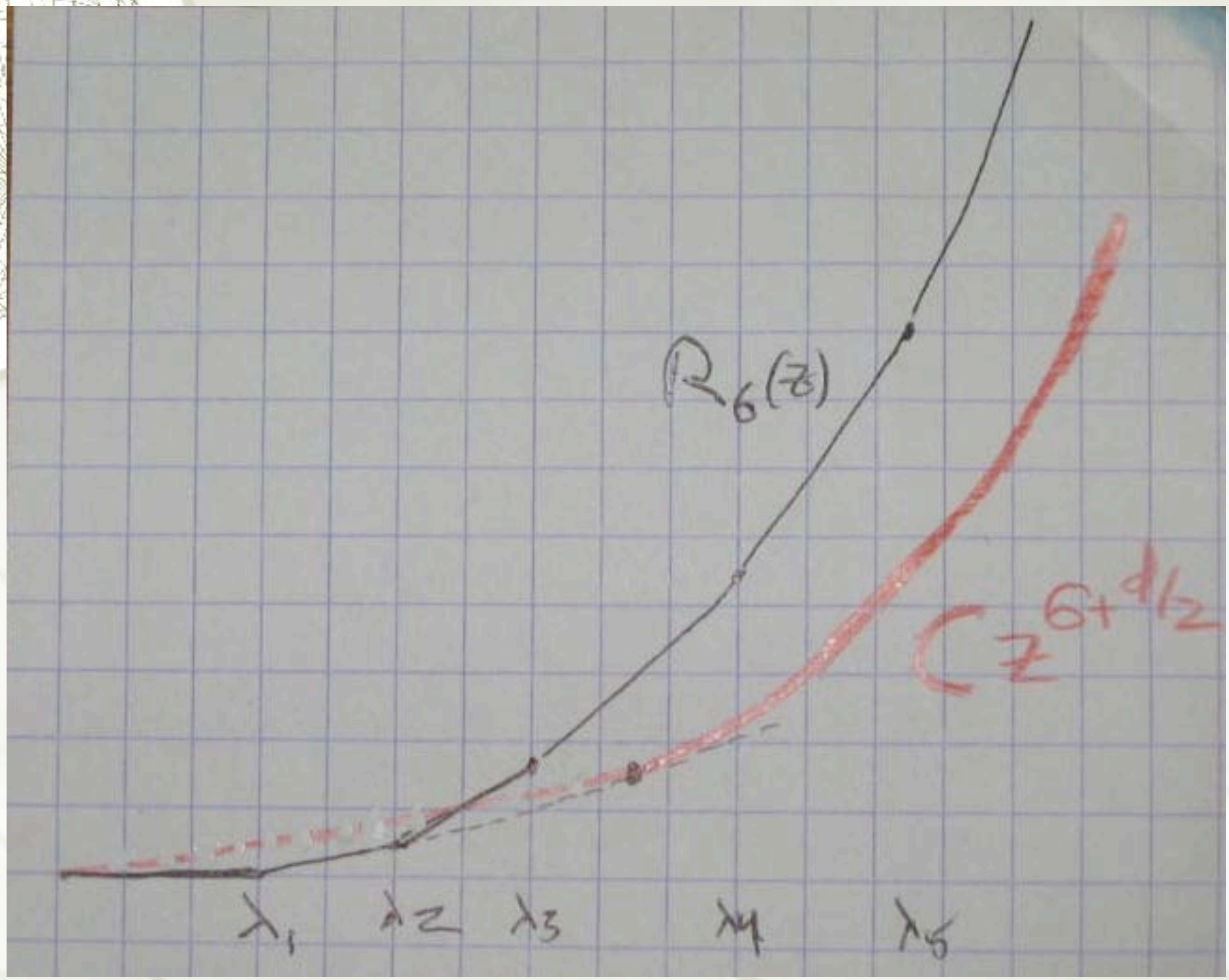

R6(z)
Cz6+d/2
λ1
λ2
λ3
λ4
λ5

Riesz means

$R_\sigma(z) := \sum (z-\lambda_k)_+^\sigma$ for $\sigma > 0$.

How is this related to *moments* of eigenvalues,

$$\sum \lambda_k^\tau$$

or, equivalently, to averages such as

$$\overline{\lambda_k} := \frac{1}{k} \sum_{\ell \leq k} \lambda_\ell$$

$$\overline{\lambda_j^2} := \frac{1}{j} \sum_{\ell \leq j} \lambda_\ell^2 \qquad ?$$

Legendre transform

Legendre transform

$$\mathcal{L}[f](w) := \sup_z \{wz - f(z)\}$$

Note: If g(z) ≥ f(z), then $\mathcal{L}[g](w) \leq \mathcal{L}[f](w)$.

Order reverses.

Legendre transform

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$$\mathcal{L}[f](w) := \sup_z \{wz - f(z)\}$$

$$R_1(z) \geq \frac{4\, d^{\frac{d}{2}}}{(d+4)^{1+\frac{d}{2}}} \lambda_1^{-\frac{d}{2}} z^{1+\frac{d}{2}}$$

becomes ….

Legendre transform

$$(w-[w])\,\lambda_{[w]+1}+[w]\,\overline{\lambda_{[w]}}\le\frac{2}{j^{\frac{2}{d}}}\left(\frac{1+\frac{d}{4}}{1+\frac{d}{2}}\right)^{1+\frac{2}{d}}\overline{\lambda_j}\;w^{1+\frac{2}{d}}$$

Meanwhile, for any … such that on the left side of (3.2), $k-1$ … et w approach k from below, we obtain fro…

$$\overline{\lambda_k}/\overline{\lambda_j}\le 2\left(\frac{1+\frac{d}{4}}{1+\frac{d}{2}}\right)^{1+\frac{2}{d}}\left(\frac{k}{j}\right)^{\frac{2}{d}}$$

$$\lambda_k+(k-1)\overline{\lambda_{k-1}}\le\frac{2}{j^{\frac{2}{d}}}\left(\frac{1+\frac{d}{4}}{1+\frac{d}{2}}\right)^{1+\frac{2}{d}}\overline{\lambda_j}k^{1+\frac{2}{d}}.$$

Which simplifies to

$$\overline{\lambda_k}/\overline{\lambda_j}\le 2\left(\frac{1+\frac{d}{4}}{1+\frac{d}{2}}\right)^{1+\frac{2}{d}}\left(\frac{k}{j}\right)^{\frac{2}{d}}$$

On a (hyper) surface, what object is most like the Laplacian?

(Δ = the good old flat scalar Laplacian of Laplace)

Answer #2 (The nanoanswer):

$$- \Delta_{LB} + q,$$

$$q(\mathbf{x}) = \frac{1}{4}\left(\sum_{j=1}^{\nu} \kappa_j\right)^2 - \frac{1}{2} \sum_{j=1}^{\nu} \kappa_j^2.$$

d=1, $q = -\kappa^2/4 \leq 0$ d=2, $q = - (\kappa_1-\kappa_2)^2/4 \leq 0$

Heisenberg's Answer
(if he had thought about it)

$$q(\mathbf{x}) := \frac{1}{4}\left(\sum_{j=1}^{d} \kappa_j\right)^2$$

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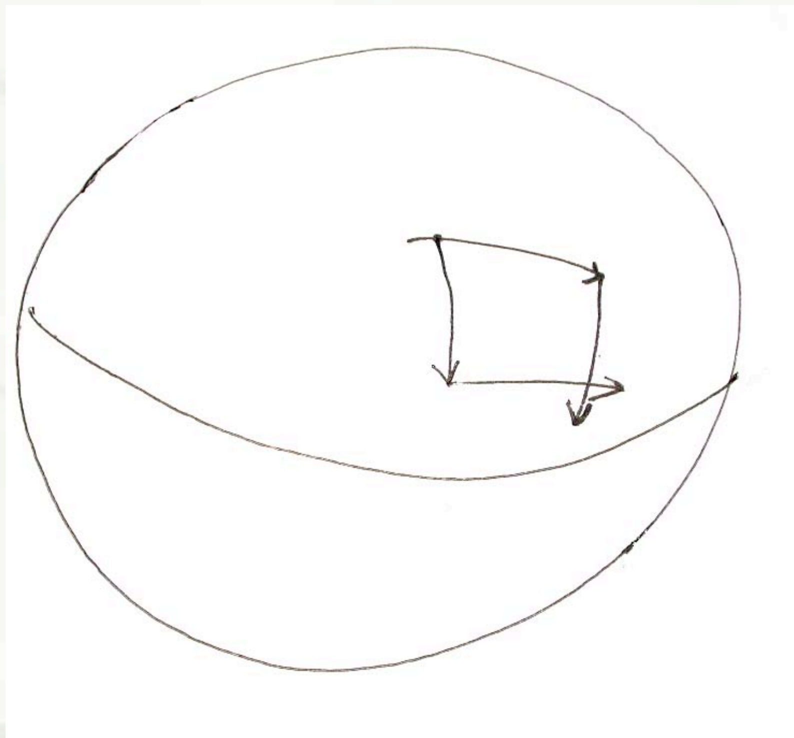
(if he had thought about it)

$$q(\mathbf{x}) := \frac{1}{4}\left(\sum_{j=1}^{d} \kappa_j\right)^2$$

Note: $q(\mathbf{x}) \geq 0$!

Commutators: [A,B] := AB-BA

3. Curvature is the effect that motions do not commute:



Commutators: [A,B] := AB-BA

3a. The equations of space curves are commutators:

$$\frac{d\mathbf{x}}{ds} = \mathbf{t}$$

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}$$

Commutators: [A,B] := AB-BA

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$$\left[\frac{d}{ds}, \mathbf{x}\right] = \mathbf{t}$$

$$\left[\frac{d}{ds}, \mathbf{t}\right] = \kappa \mathbf{n}$$

Note: curvature is defined by a second commutator

The Serret-Frenet equations as commutator relations:

$$[H, X_m] = -\frac{d^2 X_m}{ds^2} - 2\,\frac{dX_m}{ds}\frac{d}{ds} = -\kappa n_m - 2t_m\frac{d}{ds}, \qquad (2.2)$$

$$[X_m\,[H, X_m]] = 2t_m^2. \qquad (2.3)$$

Proposition 2.1 *Let M be a smooth curve in $\mathbb{R}^\nu, \nu = 2$ or 3. Then for* $H = -\dfrac{d^2}{ds^2} + \mathrm{V(s)}$ *and* $\varphi \in W_0^1(M)$,

$$\sum_{m=0}^{\mathrm{d}} \left\| [H, X_m]\, \varphi \right\|^2 = 4 \int_M \left(\left| \frac{d\varphi}{ds} \right|^2 + \frac{\kappa^2}{4} \left| \varphi \right|^2 \right) ds.$$

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$$\sum_{m=0}^{d} \|[H, X_m]\,\varphi\|^2 = 4\int_M \left(\left|\frac{d\varphi}{ds}\right|^2 + \frac{\kappa^2}{4}|\varphi|^2\right) ds.$$

Proof. By closure it may be assumed that $\varphi \in C_c^\infty(M)$. Apply (2.2) to φ and square the result, to obtain

$$4\left(t_m^2\left(\frac{d\varphi}{ds}\right)^2 + \frac{1}{4}\kappa^2 n_m^2 \varphi^2 + \frac{1}{2}\kappa n_m t_m \varphi \frac{d\varphi}{ds}\right).$$

Sum on m and integrate. QED

Corollary 2.2 *Let M be as in* Proposition 2.1 *and suppose that H is a Schrödinger Hamiltonian with a bounded measurable potential $V(s)$. Then*

$$\Gamma \leq 4 \int_M \left(\left(\frac{du_1}{ds} \right)^2 + \frac{\kappa^2}{4} u_1^2 \right) ds. \tag{2.5}$$

$\Gamma := \lambda_2 - \lambda_1$

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Furthermore, if H is of the form

$$H_g := -\frac{d^2}{ds^2} + g\kappa^2,$$

then

$$\Gamma \leq \max\left(4, \frac{1}{g}\right) \lambda_1. \tag{2.6}$$

Equivalently, the universal ratio bound

$$\frac{\lambda_2}{\lambda_1} \leq \max\left(5, 1 + \frac{1}{g}\right)$$

holds. This bound is sharp for $0 < g \leq \frac{1}{4}$.

Bound is sharp for the circle:

$$\frac{\lambda_2}{\lambda_1} = \frac{4\pi^2(1+g)}{4\pi^2 g} = 1 + \frac{1}{g}.$$

Gaps bounds and spectral identities for (hyper) surfaces

Let M be a d-dimensional manifold immersed in $\mathbb{R}^{d+1}$.

Theorem 3.1 *Let H be a Schrödinger operator on M with a bounded potential, i.e.,*

$$H = -\Delta + V, \tag{3.1}$$

where V is a bounded, measurable, real-valued function on M. If M has a boundary, Dirichlet conditions are imposed (in the weak sense that H is defined as the Friedrichs extension from $C_c^\infty(M)$). Then

$$\begin{aligned}\Gamma(H) &\le \frac{1}{d}\int_M \left(4\left|\nabla_{||}u_1\right|^2 + h^2 u_1^2\right) dVol \\ &= \frac{4}{d}\left\langle u_1, \left(-\Delta + \frac{h^2}{4}\right) u_1 \right\rangle.\end{aligned} \tag{3.2}$$

Here h is the sum of the principal curvatures.

Corollary 3.2 *Let H be as in (3.1) and define* $\delta := \sup_M \left(\frac{h^2}{4} - V\right)$. *Then*

$$\Gamma(H) \leq \frac{4}{d}\left(\lambda_1 + \delta\right).$$

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In particular, for the Laplacian, the first nontrivial eigenvalue (in this notation λ_2, with $\lambda_1 = 0$) is bounded above by the square of the mean curvature. (Reilly's inequality)

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In particular, for the Laplacian, the first nontrivial eigenvalue (in this notation λ_2, with $\lambda_1 = 0$) is bounded above by the square of the mean curvature.

Actually, each λ_k is bounded above by a simple universal constant times $\|h\|_\infty^2$.

A further corollary is an isoperimetric spectral theorem for operators of the form H_g from (1.10):

Corollary 3.3 *Let H_g be defined on M, a d-dimensional manifold smoothly immersed in $\mathbb{R}^{d+1}$. Then the eigenvalues satisfy*

$$\lambda_2 - \lambda_1 \leq \frac{4\sigma\lambda_1}{d}, \tag{3.7}$$

where $\sigma = \max\left(1, \frac{1}{4g}\right).$

Bound is sharp for the sphere:

$$\lambda_1 = gd^2, \quad \lambda_2 = gd^2 + d$$

$$d = \lambda_2 - \lambda_1 \leq \left(\frac{gd^2}{gd}\right) = d.$$

THE END