Geometric lower bounds for the spectrum of elliptic PDEs with Dirichlet conditions in part

Evans M. Harrell II*

Abstract

An extension of the lower-bound lemma of Boggio is given for the weak forms of certain elliptic operators, which are in general nonlinear and have partially Dirichlet and partially Neumann boundary conditions. Its consequences and those of an adapted Hardy inequality for the location of the bottom of the spectrum are explored in corollaries wherein a variety of assumptions are placed on the shape of the Dirichlet and Neumann boundaries.

* harrell@math.gatech.edu, School of Mathematics,Georgia Tech, Atlanta, GA 30332-0160, USA. This work was supported by NSF grant DMS-0204059.

I. Introduction

E.B. Davies has promoted the use of operator and quadratic–form inequalities to obtain spectral information about Laplacians and Schrödinger operators. One of the goals of his analysis has been to understand the effect of the shape of the boundary on eigenvalues and the eigenfunctions. The present note may be viewed as a scholium to parts of the first chapter of [Dav89], where linear elliptic operators are bounded below, in the weak sense, by functions related to the shape of the boundary of a domain. The aim is the elaboration of lower bounds of similar kinds, given two complicating features: boundary conditions other than of Dirichlet type, and nonlinearities in the principal part of the operators.

As has been pointed out in [FHT99], a precursor to many inequalities used for lower bounds in the spectral theory of elliptic differential equations is to be found a 1907 article by T. Boggio [Bog07]. This direct consequence of the Gauß–Green theorem was interpreted in [FHT99] as a variational principle for the Laplacian and its nonlinear generalisation the p–Laplacian, on bounded domains with vanishing Dirichlet conditions on the boundary. Lemma I.1 below, the main tool used in this article, likewise applies to a family of nonlinear elliptic operators, and non–Dirichlet boundary conditions are allowed. No great originality is claimed for Lemma I.1: The idea is still close to that of [Bog07], and it is likely that an equivalent result could be extracted from what has been reported in [Maz81] or [OpKu90]. It is, however, stated in a form allowing geometric spectral results to be extracted easily, with attention to non–Dirichlet boundary portions. The short proof is given so that the exposition is self–contained.

The great majority of lower bounds to the spectrum of elliptic partial differential equations suppose Dirichlet conditions throughout any boundary that is present. One reason for this is that if Neumann conditions occur, the spectrum of the Laplacian is highly unstable with respect to perturbations of the boundary. An arbitrarily small perturbation of the Neumann part of the boundary may cause the lowest eigenvalue to decrease arbitrarily close to 0. Intuitively, this is possible because the lowest eigenvalue of any domain with pure Neumann conditions is 0, so if a small such domain is weakly coupled to Ω via a thin neck or partitions, then the lowest eigenvalue will remain tiny. The instability of Neumann spectra has been extensively investigated, and instructive "rooms and passages" models date from Courant and Hilbert: See [CoHi37], [Maz73], [Fra79], [EvHa89],[HSS91],[Arr95], [BuDa02] among many treatments of the Neumann spectrum.

There is evidence that if assumptions on the Neumann part of the boundary prevent its near–isolation, then the effect on the lowest eigenvalue will be more moderate. For instance, in [BuDa02], a continuous perturbation theory is established for purely Neumann Laplacians with respect to uniformly Hölder perturbations. Section II of this article includes lower bounds controlling the downward shift in the fundamental eigenvalue upon non–perturbative enlargement of the domain.

It will be assumed throughout that Ω is a bounded domain in $\mathbb{R}^d$, the boundary of which is sufficiently regular that Green's formula is valid for suitable test functions and that the outward normal vector, denoted ν, exists a.e. at any part of the boundary where conditions other than vanishing Dirichlet are imposed. Regularity depends on the validity of certain Sobolev embeddings. Here each domain Ω will be assumed to have a boundary consisting of two pieces each of positive $d-1$–dimensional Lebesgue measure, denoted

$\partial\Omega_D$ and $\partial\Omega_N := \partial\Omega \setminus \partial\Omega_N$. For the purposes of this article it suffices to suppose that the boundary is of "class C" (see [EdEv87], p. 244, 248), and that the boundary is C^1 a.e. with respect to the surface Lebesgue measure. Such a domain will be termed *regular*. (Regularity is not actually required on $\partial\Omega_D$, while on $\partial\Omega_N$ usefully wider conditions of regularity may possibly be gleaned from [Ken94]. The widest conditions for the boundary shall not be pursued here.) An adapted set of test functions $\mathcal{D}(\Omega, \Omega_D)$ can be defined as consisting of the restrictions to Ω of $C_c^\infty\left(\mathbb{R}^d \setminus \partial\Omega_D\right)$.

Although the notation $\partial\Omega_N$ is introduced with Neumann boundary conditions in mind, no such assumption is made. Only the Dirichlet portion of the boundary is actually subject to (weakly defined) boundary conditions. This article is concerned solely with functionals on $\mathcal{D}$, without investigating the domains of definition of the elliptic operators that the functionals may define.

Lemma I.1. *Suppose that Ω is regular, $1 < p < \infty$, $\mathbf{Q}$ is a real–valued absolutely continuous vector field, and A is a real–valued absolutely continuous tensor field of type $\mathcal{T}_1^1$.* (In elementary terms, A is a $d \times d$ real matrix–valued function acting on $\mathbf{Q}$.) *Suppose further that $\nu \cdot A^T\mathbf{Q} \le 0$ a.e. on Ω_N. Then for all $\zeta \in \mathcal{D}(\Omega, \Omega_D)$,*

$$\int_\Omega |A\nabla\zeta|^p dx \;\ge\; \int_\Omega \left\{\operatorname{div}\left(A^T\mathbf{Q}\right) \;-\; (p-1)|\mathbf{Q}|^{p'}\right\} |\zeta|^p \; dx, \tag{1.1}$$

where p' is the dual index, $p' = \frac{p}{p-1}$.

Alternatively, let a and q be absolutely continuous functions on Ω and let $\mathbf{u}$ be a fixed unit vector such that $aq\mathbf{u} \cdot \nu \le 0$ a.e. on $\partial\Omega_N$. Then for all $\zeta \in \mathcal{D}(\Omega, \Omega_D)$,

$$\int_\Omega |a\mathbf{u} \cdot \nabla\zeta|^p \, dx \ge \int \left\{\mathbf{u} \cdot \nabla(aq) - (p-1)q^{p'}\right\} |\zeta|^p dx. \tag{1.2}$$

Proof. We calculate following [FHT99] with the divergence theorem:

$$\begin{aligned} 0 \le -\int_{\partial\Omega} \nu \cdot \left(A^T\mathbf{Q}|\zeta|^p\right) dS &= -\int_\Omega \operatorname{div}\left(A^T\mathbf{Q}|\zeta|^p\right) dx \\ &= -\int_\Omega (\operatorname{div} A^T\mathbf{Q})|\zeta|^p dx - p\int_\Omega |\zeta|^{p-2}\zeta\mathbf{Q} \cdot A\nabla\zeta dx. \end{aligned}$$

With $\mathbf{w} = |\zeta|^{p-2}\zeta\mathbf{Q}$, the arithmetic–geometric mean inequality [HLP34] gives

$$|\,(A\nabla\zeta) \cdot \mathbf{w}| \le \frac{1}{p}\,|A\nabla\zeta|^p + \frac{1}{p'}\,|\mathbf{w}|^{p'} = \frac{1}{p}\,|A\nabla\zeta|^p + \frac{1}{p'}\,|\zeta|^p|\mathbf{Q}|^{p'}.$$

Collecting terms,

$$\int_\Omega |A\nabla\zeta|^p dx \ge \int_\Omega \left(\operatorname{div}\left(A^T\mathbf{Q}\right) - (p-1)\;|\mathbf{Q}|^{p'}\right) |\zeta|^p dx,$$

which is (1.1). The proof of (1.2) is strictly analogous, beginning with the Gauß–Green formula in the version

$$0 \le -\int_{\partial\Omega} (aq|\zeta|^p)\,\nu^i dS = -\int_{\partial\Omega} \frac{\partial\,(aq|\zeta|^p)}{\partial x_i} dx$$

for a Euclidean coordinate x_i (e.g., [Eva98]). The result is

$$\int_\Omega \left| a \frac{\partial \zeta}{\partial x_i} \right|^p dx \ge \int \left\{ \frac{\partial (aq)}{\partial x_i} - (p-1) q^{p'} \right\} |\zeta|^p dx,$$

which when written without coordinates becomes (1.2). □

Remarks

1. Inequality (1.1) is well–known in the case $p = 2$ with Dirichlet boundary conditions on the entire boundary, in the form obtained with

$$\mathbf{Q}_{std} := -A\nabla \log(\Phi),$$

for a suitable function $\Phi > 0$ on Ω. For $p \neq 2$ the analogous choice is

$$\mathbf{Q}_{std} = -\frac{|A\nabla\Phi|^{p-2}\,A\nabla\Phi}{\Phi^{p-1}}.$$

A simple calculation shows that for this choice Lemma I.1 becomes

$$\int_\Omega |A\nabla\zeta|^p dx \;\ge\; \int_\Omega \frac{-\mathrm{div}\left(|A\nabla\Phi|^{p-2}\,A^T A\nabla\Phi\right)}{\Phi^{p-1}} |\zeta|^p dx, \tag{1.3}$$

which for $p = 2$ is essentially Theorem 1.5.12 of [Dav89]. Let us refer to this as the *standard form* of Inequality (1.1). It is all the more familiar for the linear Dirichlet Laplacian, i.e., with A taken as the identity and $p = 2$, in which case it becomes the well-known bound

$$-\Delta \ge \frac{-\Delta\Phi}{\Phi},$$

which is often attributed to Barta or Duffin, who, however, published much later than Boggio [Bog07]. (In the not widely accessible [Bog07], Boggio used the Gauß-Green Theorem to prove the case of Lemma I.1 with $d = 2 = p$, pure Dirichlet, $A =$ the identity.) Lemma I.1 is truly more general than the standard form, which corresponds to a vector field $\mathbf{Q}$ restricted to be A times something irrotational.

2. Similarly, if

$$q_{std} := \frac{-\left| a \frac{\partial\Phi}{\partial x_i} \right|^{p-2} \frac{\partial\Phi}{\partial x_i}}{\Phi^{p-1}}$$

is inserted into (1.2), the result is

$$\int_\Omega \left| a \frac{\partial \zeta}{\partial x_i} \right|^p dx \ge \int_\Omega \frac{-\frac{\partial}{\partial x_i}\left(\left| a \frac{\partial\Phi}{\partial x_i} \right|^{p-2} \frac{\partial\Phi}{\partial x_i} \right)}{\Phi^{p-1}} |\zeta|^p dx, \tag{1.4}$$

provided that $aq\nu^i \leq 0$ a.e. on $\partial\Omega_N$. The expression in the numerator of the lower bound (1.4) is a one-dimensional p–Laplacian.

3. Lemma I.1 may be thought of as a lower–bound variational principle for the bottom of the spectrum of the operators whose energy forms are as in the left side of (1.1), i.e.,

$$H\zeta := -\mathrm{div}\left(|A\nabla\zeta|^{p-2} A^T A\nabla\zeta\right). \tag{1.5}$$

If $\Phi > 0$ is the fundamental eigenfunction of H, that is, $H\Phi = \lambda_1\Phi^{p-1}$, then the lower bound to the spectrum of H thus obtained reduces to λ_1. In this sense the inequality is optimal. While it must be conceded that the inequality is thus arguably not better in theory than the standard form for the purpose of finding lower bounds, it has two practical advantages:
   a) It is relatively easy to construct a vector field with positive divergence to cover a domain, the only material restriction being that it be incoming on $\partial\Omega_N$.
   b) The nonlinearity of the bound can be exploited. If *any* vector field can be constructed on Ω with the necessary boundary behavior and strictly positive divergence, then by scaling $\mathbf{Q} \to t\mathbf{Q}$ a strictly positive lower bound will be obtained for sufficiently small $t > 0$.
4. An extension from Euclidean domains to subdomains of orientable smooth manifolds is straightforward.

In the case where A is the identity tensor, H corresponds to the p–Laplacian, denoted $-\Delta_p$. It is known that the minimum of the functional

$$\int_\Omega |\nabla\zeta|^p \, dx,$$

for $\|\zeta\|_p = 1$ and Dirichlet boundary conditions, corresponds to an eigenvalue λ_1 of $-\Delta_p$ Many other familiar facts for the linear case $p = 2$ carry over, such as that the minimiser u_1 exists in the appropriate Sobolev space and is positive on Ω. The second eigenvalue of (1.5) is also an inflection point of (1.5) and can thus be characterised variationally. However, the theory of the higher eigenvalues of $-\Delta_p$ remains murky when $p \neq 2$. Detailed analysis of the p-Laplacian will not be entered into here, as the focus will be entirely on lower bounds for its energy form H. (The spectral theory of linear elliptic differential operators is presented, for example, in [EdEv87], [Dav95], and a useful review of the spectral theory of the p–Laplacian is to be found in [DKN96, DKT99].)

In the following section particular vector fields $\mathbf{Q}$ will be chosen responding to assumptions about the shape of the domain Ω.

II. Spectral bounds with Dirichlet conditions on subsets of the boundary

The aim of this section is to find lower bounds to the spectrum in terms of the shapes of $\partial\Omega_D$ and $\partial\Omega_N$. For simplicity it will be assumed henceforth that A is the identity tensor, i.e., the case of the p–Laplacian. No material difficulty would arise if A were retained.

In essence this section consists of a selection of examples of the use of Lemma I.1, organized as a list of corollaries.

Definition II.1. Let $\lambda_1(\Omega, \partial\Omega_D, p)$ denote the infimum for $\zeta \in \mathcal{D}(\Omega, \partial\Omega_D)$ (and not identically zero) of

$$E_p(\zeta) := \left(\frac{\int_\Omega |\nabla\zeta|^p}{\int_\Omega |\zeta|^p} \right).$$

The quantity $\lambda_1(\Omega, \partial\Omega_D, p)$ will be referred to as the *fundamental eigenvalue of the p–Laplacian* with respect to $\Omega, \partial\Omega_D$.

As remarked in the introduction, eigenvalues can be rather unstable with respect to Neumann conditions on the boundary. In the purely Dirichlet case the principle of domain–monotonicity is familiar: If $\Omega \subset \Upsilon$, then $\lambda_1(\Omega, \partial\Omega, p) \geq \lambda_1(\Upsilon, \partial\Upsilon, p)$. There is no such general principle in the presence of Neumann conditions. Nonetheless, Lemma I.1 implies the following:

Corollary II.2 – Restricted Monotonicity Principle. *Let Ω and Υ be regular domains such that $\Omega \subset \Upsilon$. Let $u_1(\Upsilon, \partial\Upsilon_D, p)$ be a fundamental eigenfunction for the p–Laplacian with respect to Υ, Υ_D, and suppose that $u_1(\Upsilon, \partial\Upsilon_D, p) > 0$ on Ω. If $\nu \cdot \nabla u_1(\Upsilon, \partial\Upsilon_D, p) \geq 0$ a.e. on $\partial\Omega_N$, then $\lambda_1(\Omega, \partial\Omega_D, p) \geq \lambda_1(\Upsilon, \partial\Upsilon_D, p)$.*

Remark. Although $u_1(\Upsilon, \partial\Upsilon_D, p)$ may be taken positive with no loss of generality when, for example, p=2 and the boundary conditions are purely of Dirichlet and Neumann type, this is not automatic in the widest circumstances.

Proof. The assumptions allow the standard form (1.3) of the lower bound with the choice $\Phi = u_1(\Upsilon, \partial\Upsilon_D, p)$. In this case the right side of (1.3) reduces to $\lambda_1(\Upsilon, \partial\Upsilon_D, p)$. □

An immediate consequence is that if the Neumann boundary is a graph with respect to a hyperplane, there is an analogue of the elementary outradius bound for the Dirichlet problem. There results a crude "box bound" in terms of $\mu_{I,p} :=$ the fundamental eigenvalue of the one–dimensional p–Laplacian $-\Delta_p^{(1)}$ on the unit interval, with Dirichlet conditions at one end and Neumann at the other. (Thus $\mu_{I,2} = \frac{\pi^2}{4}$.)

Corollary II.3 – Box Bound. *Let Ω be a regular domain such that $\partial\Omega_N$ is the graph of a function on a subset of $\{\mathbf{x} : x_i = 0\}$, with $\nu \cdot \mathbf{e}_i \leq 0$ a.e. on $\partial\Omega_N$ For $L := \sup(x_i : \mathbf{x} \in \Omega) - inf\,(x_i : \mathbf{x} \in \Omega)$,*

$$\lambda_1(\Omega, \partial\Omega_D, p) \geq \frac{\mu_{I,p}}{L^p}.$$

Proof. Since the fundamental eigenfunction $u_{1,p}(x)$ of the one–dimensional p–Laplacian with Dirichlet conditions at one end and Neumann at the other is positive and monotonic (due to the maximum principle), we may choose $a = 1$, $\Phi = u_{1,p}$ in (1.4), yielding the result. □

Less crude bounds involving the shape of Ω can be obtained with more sophisticated comparisons. Recall that in [Dav89], Davies exploited Hardy's one–dimensional inequality [Har20] [HLP34] to produce lower bounds in the quadratic–form sense of the type

$$-\Delta \geq \frac{C}{(m(\mathbf{x}))^2},$$

where m is an averaged distance to the boundary, supposed entirely Dirichlet. These bounds extend almost immediately to the case of the p–Laplacian and boundaries that are only partly Dirichlet, with certain restrictions:

Definition II.4. Given a regular domain Ω, the *Dirichlet boundary sector* of $\mathbf{x}, \partial\Omega_D$, denoted $S(\mathbf{x}, \partial\Omega_D)$, is the union of all line segments contained in Ω joining $\mathbf{x}$ to $\partial\Omega_D$. Following [Dav89], an averaged distance to the Dirichlet boundary is given by

$$\frac{1}{(m(\mathbf{x}))^p} := \frac{1}{\omega_d} \int_K \frac{dS(\mathbf{u})}{(d_{\mathbf{u}}(\mathbf{x}))^p}$$

where the measure $dS(\mathbf{u})$ is the Lebesgue measure on the unit sphere S^{d-1}, ω_d is the total measure of S^{d-1}, $K := S(\mathbf{x}, \partial\Omega_D) \cap S^{d-1}$, and $d_{\mathbf{u}}$ is the length of a line segment joining $\mathbf{x}$ to $\partial\Omega_D$.

Note that the Dirichlet boundary sector of $\mathbf{x}$ may be vacuous, and $\frac{1}{m}$ may be zero.

Lemma II.5. *Let Ω be a regular domain and $\zeta \in \mathcal{D}(\Omega, \Omega_D)$. Then for any $1 < p \leq 2$,*

$$\int_\Omega |\nabla\zeta|_2^p \, dx \geq \int_\Omega |\nabla\zeta|_p^p \, dx \geq dp^{-p}(p-1)^{p-2} \int_\Omega \left| \frac{\zeta(\mathbf{x})}{m(\mathbf{x})} \right|^p dx.$$

For any $1 \leq p < \infty$

$$\int_\Omega |\nabla\zeta|_2^p \, dx \geq d^{\frac{2-p}{2}} \int_\Omega |\nabla\zeta|_p^p \, dx \geq d^{\frac{4-p}{2}} p^{-p}(p-1)^{p-2} \int_\Omega \left| \frac{\zeta(\mathbf{x})}{m(\mathbf{x})} \right|^p dx.$$

With the one–dimensional L^p Hardy inequality (i.e., [OpKu90] Theorem 1.14 with $p = q$, $v = 1$, and $w(x) = (x-a)^{-p}$, itself derivable from lemma I.1), the proof of the lemma is almost exactly as for [Dav89], Theorem 1.5.3, and therefore details will be omitted. The additional factors are optimal constants relating the 2–norm and the p–norms on $\mathbb{R}^d$, as derived, for instance, in [FHT99]. For brevity, define the constant $C_D(p, d)$ so that Lemma II.5 reads

$$\int_\Omega |\nabla\zeta|_2^p \, dx \geq C_D(p, d) \int_\Omega \left| \frac{\zeta(\mathbf{x})}{m(\mathbf{x})} \right|^p dx$$

for all p, d.

Suppose that $\partial\Omega_N$ is starlike with respect to an exterior point, at which the origin will be placed. The unit radial vector pointing away from the origin will as usual be denoted $\mathbf{e}_r$. It will now be shown that there is a sort of multidimensional Hardy inequality with respect to a distance function that is intuitively the lesser of a multiple of $|\mathbf{x}|$ and the distance to the Dirichlet part of the boundary. The precise version of this statement is expressed by (2.2) in Corollary II.6 and (2.3) in Corollary II.7. Versions of these corollaries, in the linear case $p = 2$, have been previously reported, though not published, by the author since [Har93]. They imply, for example, that it is possible to have a fractal $\partial\Omega_N$ without the collapse of the bottom of the spectrum to 0.

Corollary II.6. *Let $1 < p < d$, and suppose that the origin is exterior to Ω, that $\partial\Omega_N$ is starlike with respect to the origin, and that $\nu \cdot \mathbf{e}_r \leq 0$ a.e. on $\partial\Omega_N$. Then for all $\zeta \in \mathcal{D}(\Omega, \partial\Omega_D, p)$,*

$$\int_\Omega |\nabla\zeta|^p \, dx \geq \left(\frac{d-p}{p}\right)^p \int_\Omega \left|\frac{\zeta}{|\mathbf{x}|}\right|^p dx. \tag{2.1}$$

Consequently,

$$\lambda_1(\Omega, \partial\Omega_D, p) \geq \left(\frac{d-p}{p \inf\{|\mathbf{x}| : \mathbf{x} \notin \Omega\}}\right)^p.$$

Moreover, for any $\gamma, 0 \leq \gamma \leq 1$,

$$\lambda_1(\Omega, \partial\Omega_D, p) \geq \inf_{\mathbf{x}\in\Omega} \left(\gamma \left(\frac{d-p}{p|\mathbf{x}|}\right)^p + (1-\gamma)\frac{C_D(p,d)}{m(\mathbf{x})^p}\right). \tag{2.2}$$

Proof. Eq. (2.1) results from the choice $\mathbf{Q} = \left(\frac{d-p}{p}\right)^{p-1} \frac{\mathbf{x}}{|\mathbf{x}|^p}$ in Lemma I.1 and a calculation. The next statement is the immediate lower bound by replacing a factor in the integrand by its infimum. Statement (2.2) is obtained in the same way from the weighted average of the lower bounds of Lemma II.5 and (2.1). □

When $p > d$, the origin needs to be placed on the other side of $\partial\Omega_N$:

Corollary II.7. *Let $d < p < \infty$, and suppose that the origin is exterior to Ω, that $\partial\Omega_N$ is starlike with respect to the origin, and that $\nu \cdot \mathbf{e}_r \geq 0$ a.e. on $\partial\Omega_N$. Then for all $\zeta \in \mathcal{D}(\Omega, \partial\Omega_D, p)$,*

$$\int_\Omega |\nabla\zeta|^p \, dx \geq \left(\frac{p-d}{p}\right)^p \int_\Omega \left|\frac{\zeta}{|\mathbf{x}|}\right|^p dx. \tag{2.3}$$

Consequently,

$$\lambda_1(\Omega, \partial\Omega_D, p) \geq \left(\frac{p-d}{p \inf\{|\mathbf{x}| : \mathbf{x} \notin \Omega\}}\right)^p.$$

Moreover, for any $\gamma, 0 \leq \gamma \leq 1$,

$$\lambda_1(\Omega, \partial\Omega_D, p) \geq \inf_{\mathbf{x}\in\Omega} \left(\gamma \left(\frac{p-d}{p|\mathbf{x}|}\right)^p + (1-\gamma)\frac{C_D(p,d)}{m(\mathbf{x})^p}\right). \tag{2.4}$$

Proof. Eq. (2.3) results from the choice $\mathbf{Q} = -\left(\frac{p-d}{p}\right)^{p-1} \frac{\mathbf{x}}{|\mathbf{x}|^p}$ in Lemma I.1 and a calculation. The other statements follow as in Corollary II.6. □

For the missing case $p = d$, a comparison can be made with annular domains, i.e., the region between two concentric circles or spheres.

Definition II.8. Let $\alpha(r, R, p, d)$ denote the lowest eigenvalue of the p–Laplacian on $A := \{\mathbf{x} : r < |\mathbf{x}| < R\}$, with Dirichlet boundary conditions imposed where $|\mathbf{x}| = R$ and Neumann boundary conditions imposed where $|\mathbf{x}| = r$. Similarly, let $\beta(r, R, p, d)$

denote the lowest eigenvalue of the p–Laplacian on A, with Neumann boundary conditions imposed where $|\mathbf{x}| = R$ and Dirichlet boundary conditions imposed where $|\mathbf{x}| = r$.

Observe that $\alpha(r, R, p, d)$ and $\beta(r, R, p, d)$ can be written in terms of explicit Bessel functions, and that for any $p > 1$ α and β are determined by ordinary differential equations, as the fundamental eigenfunctions are radial. The eigenvalues $\alpha(r, R, 2)$ and $\beta(r, R, 2)$ and their associated eigenfunctions are thus numerically accessible. The Restricted Monotonicity Principle II.2 immediately implies:

Corollary II.9 – Annulus Bound.

a) Let $p > 1$ and suppose that the origin is exterior to Ω, that $\partial\Omega_N$ is starlike with respect to the origin, and that $\nu \cdot \mathbf{e}_r \leq 0$ a.e. on $\partial\Omega_N$. Suppose further that $\Omega \subset \{\mathbf{x} : r < |\mathbf{x}| < R\}$. Then $\lambda_1(\Omega, \partial\Omega, p) \geq \alpha(r, R, p, d)$.

b) Let $p > 1$ and suppose that the origin is exterior to Ω, that $\partial\Omega_N$ is starlike with respect to the origin, and that $\nu \cdot \mathbf{e}_r \geq 0$ a.e. on $\partial\Omega_N$. Suppose further that $\Omega \subset \{\mathbf{x} : r < |\mathbf{x}| < R\}$. Then $\lambda_1(\Omega, \partial\Omega, p) \geq \beta(r, R, p, d)$.

Remark. The Annulus Bound, as well as the Box Bound II.3, can be combined with Lemma II.5 to produce lower bounds analogous to (2.2) and (2.4), taking the Dirichlet boundary into account.

Finally, let us briefly consider the effect of changing a portion of the boundary from $\subset \partial\Omega_N$ to $\subset \partial\Omega_D$ or *vice versa*. To focus the analysis, it will be supposed that Ω is convex, and that the unperturbed problem has boundary of one type, either $\partial\Omega = \partial\Omega_N$ or $\partial\Omega = \partial\Omega_D$.

The former case is easier to analyze, and indeed the perturbation theory when additional Dirichlet conditions on small subsets of positive capacity has been studied systematically, for example in [Oza81], [Oza82], [Cou95], [McG96], [McG98]. A rough non–asymptotic estimate follows from Lemma II.5:

Corollary II.10. *Let $d \geq 2$, $1 < p < \infty$, and suppose that Ω is strictly convex. Define $\epsilon(\Omega, p)$ as the minimum, for $\mathbf{x}, \mathbf{y} \in \partial\Omega$ and $\nu_{\mathbf{y}}$ of length 1 and normal to a support plane at $\mathbf{y}$, of $\frac{(\mathbf{y}-\mathbf{x})\cdot\nu_{\mathbf{y}}}{|\mathbf{y}-\mathbf{x}|^{p+d}}$.* (This is a measure of the eccentricity of Ω in relation to its diameter.) *Then*

$$\lambda_1(\Omega, \partial\Omega_D, p, d) \geq \frac{C_D(p, d)\epsilon(\Omega, p)\,|\partial\Omega_D|}{\omega_d}.$$

Proof. Strict convexity implies that a change of variable can be made in the integral in Definition II.4 of $\frac{1}{(m(\mathbf{x}))^p}$ passing from $\mathbf{u} \in K \subset S^{d-1}$ to $\mathbf{y} \in \partial\Omega_D$, as K is the projection to the unit sphere of $\partial\Omega_D$. This requires a Jacobian factor $\frac{(\mathbf{y}-\mathbf{x})\cdot\nu_{\mathbf{y}}}{|\mathbf{y}-\mathbf{x}|^{d}}$, $\nu_{\mathbf{y}}$ being the outward normal where defined. (By assumption in this article the outward normal is defined a.e., but it is convenient to let $\nu_{\mathbf{y}}$ range over the normals to support planes, for definiteness at all boundary points and so the formula will remain valid for some irregular domains.) The infimum of the transformed integrand is decreased by allowing $\mathbf{x}$ and $\mathbf{y}$ to vary independently over the closure of Ω. In that circumstance, it is easy to see that the minimum is attained for $\mathbf{x}, \mathbf{y} \in \partial\Omega$, yielding the result. □

The problem of changing a portion of a Dirichlet boundary to Neumann is more delicate than the converse. If a portion of the boundary is changed from having Dirichlet conditions to Neumann, then a positive lower bound to λ_1 can be exhibited under starlikeness conditions of Corollaries II.6, II.7, or II.9. Such bounds will, however, fail to approach the unperturbed eigenvalue as $|\partial\Omega_N| \to 0$. The perturbation theory for the introduction of Neumann boundaries has apparently been studied only under very special conditions to date (e.g., [Oza85]). A full theory will probably require a careful consideration of the asymptotic behavior of eigenfunctions near the perturbation.

References

[Arr95] J.M. Arrieta, Neumann eigenvalue problems on exterior perturbations of the domain. *J. Diff. Eq.* **118** (1995)54–103.

[Bog07] T. Boggio, Sull'equazione del moto vibratorio delle membrane elastiche, *Accad. Lincei, sci. fis.*, ser. 5a**16**(1907)386–393.

[BuDa02] V.I. Burenkov and E. B. Davies, Spectral stability of the Neumann Laplacian. *J. Diff. Eq.* **186** (2002), no. 2, 485–508.

[CoHi37] R. Courant and D. Hilbert, Methods of mathematical physics, vol. II, Interscience Publ., Wiley, New York, 1962. (Original publication 1937.)

[Cou95] G. Courtois, Spectrum of manifolds with holes, *J. Funct. Analysis* **134**(1995)194–221.

[Dav95] E.B. Davies, *Spectral theory and differential operators*, Cambridge Studies in Advanced Mathematics **42**. Cambridge: At the University Press, 1995.

[Dav89] E.B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics **92**. Cambridge: At the University Press, 1989.

[DKN96] P. Drábek, A. Kufner , and F. Nicolosi, *Nonlinear differential equations, Singular and degenerate case*, Pilsen, Czech Republic: University of West Bohemia, 1996.

[DKT99] P. Drábek, P. Krejčí, and P. Takáč, *Nonlinear differential equations*, Boca Raton, Florida: CRC Press, 1999.

[EdEv87] D.E. Edmunds and W.D. Evans, *Spectral theory and differential operators*, Oxford: Clarendon Press, 1987.

[Eva98] L.C. Evans, *Partial differential equations*, Graduate Studies in Mathematics **19**. Providence: American Mathematical Society, 1998.

[EvHa89] W.D. Evans, and D.J and Harris, On the approximation numbers of Sobolev embeddings for irregular domains, *Quarterly J. Math. Oxford* (2) **40**(1989)13–42.

[FHT99] J. Fleckinger, E.M. Harrell, II, and F. de Thélin, Boundary behavior and L^q estimates for solutions of equations containing the p–Laplacian, *Electronic J. Diff. Eq.* **1999**(1999)1–19.

[Fra79] L.E. Fraenkel, On regularity of the boundary in the theory of Sobolev spaces. *Proc. Lond. Math. Soc.* (3) **39**(1979)385–427.

[Har20] G.H. Hardy, Note on a theorem of Hilbert. *Math. Z.* **6**(1920)314–317.

[HLP34] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities.* Cambridge: At the University Press, 1959. Original publication 1934.

[Har93] E.M. Harrell II, Lecture at the Workshop on Partial Differential Equations and Fractals, Toulouse, France, May, 1993, unpublished.

[HSS91] R. Hempel, L.A. Seco, and B. Simon, The essential spectrum of Neumann Laplacians on some bounded singular domains, *J. Funct. Analysis* **102**(1991)448–483.

[Ken94] C. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics 83. Providence: American Mathematical Society, 1994.

[Maz81] V.G. Maz'ja, *Sobolev spaces.* New York: Springer, 1985. First published 1981.

[Maz73] V.G. Maz'ja, On (p, l)–capacity, embedding theorems, and the spectrum of a self-adjoint elliptic operator, *Izv. Akad. Nauk SSSR ser. Mat.* **37**(1973)356-385: *Transl. Math. USSR IZV* 7(1973)357-387.

[McG96] I. McGillivray, Capacitary estimates for Dirichlet eigenvalues, *J. Funct. Analysis* **139**(1996)244–259.

[McG98] I. McGillivray, Capacitary asymptotic expansion of the groundstate to second order, *Commun. Part. Diff. E.* **23** (1998)2219–2252

[OpKu90] B. Opic and A. Kufner, *Hardy–type inequalities,* Pitman Research Notes in Math. **219** Boston: Longman, 1990.

[Oza81] S. Ozawa, Singular variation of domains and eigenvalues of the Laplacian. *Duke Math. J.* **48**(1981)767–778.

[Oza82] S. Ozawa, An Asymptotic Formula for the Eigenvalues of the Laplacian in a Domain with a Small Hole, *Proc. Japan Acad.* **58**, ser. A (1982)5–8.

[Oza85] S. Ozawa, Asymptotic property of an eigenfunction of the Laplacian under singular variation of domains–the Neumann condition. *Osaka J. Math.* **22** (1985)639–655.