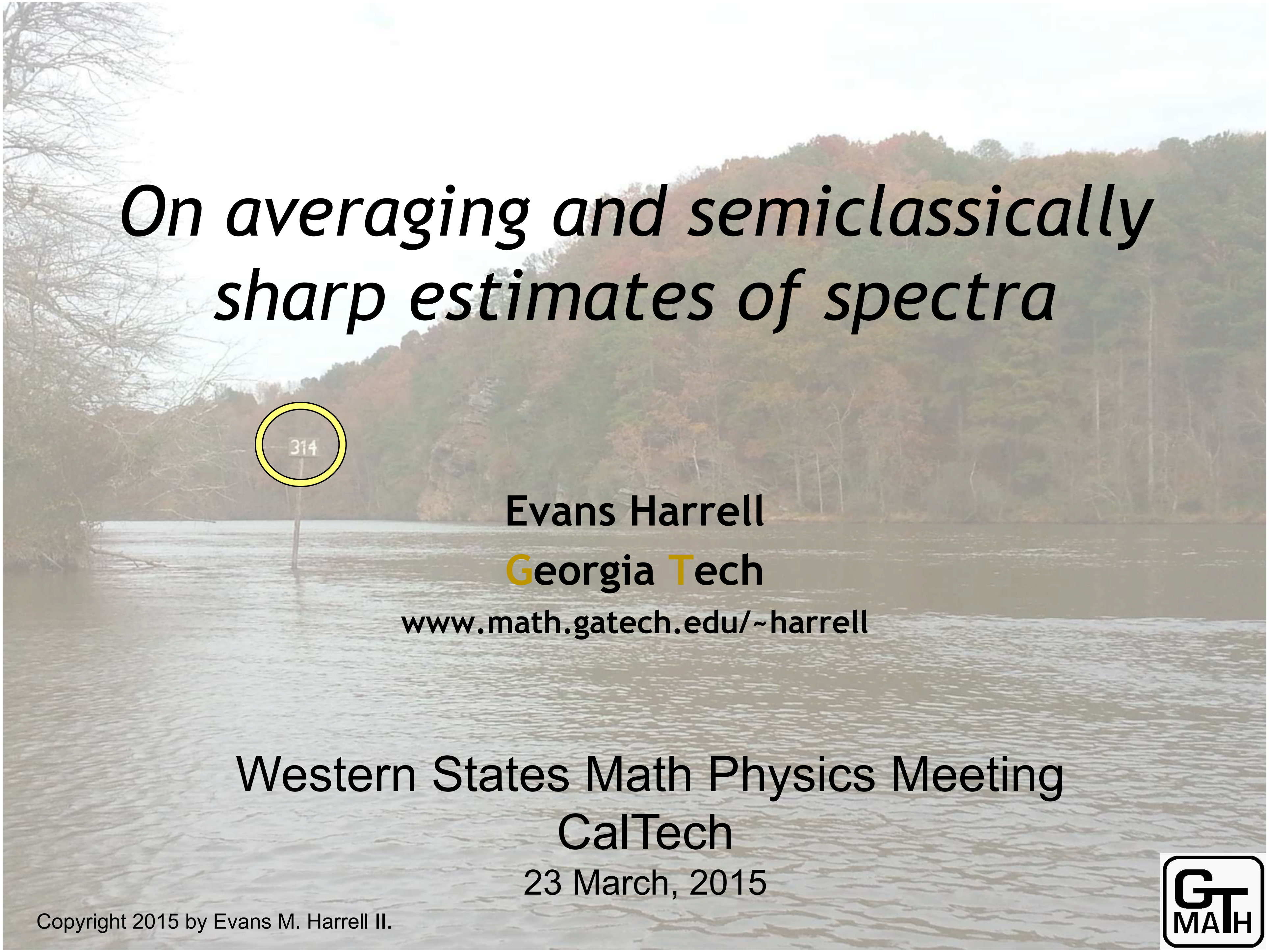

On averaging and semiclassically sharp estimates of spectra

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Abstract

- I'll present some tools for spectral analysis in which averaging enters, and show how they can be used to obtain semiclassically sharp bounds and universal control on the statistical distribution of eigenvalues. Applications will be made to sums of eigenvalues, partition functions, and other spectral quantities for a wide category of elliptic PDEs, as well as analogous operators on graphs.

This is joint work with J. Stubbe of EPFL and A. El Soufi and S. Ilias, Univ. de Tours, and in part with John Dever, GT grad student.

Sums of eigenvalues

- Suppose that you know about

$$S_k := \sum_{\ell=0}^{k-1} \mu_\ell$$

(say, upper or lower bounds). What else do you know?

Sums of eigenvalues

- With the Laplace transform,

$$\mathcal{L}\left((z-\mu_\ell)_+^\sigma\right) = \frac{\Gamma(\sigma+1)e^{-\mu_\ell t}}{t^{\sigma+1}}$$

so knowing about sums means knowing about the partition function, which in turn connects to spectral zeta functions, etc.

Karamata's theorem

Lemma 3.1 (Karamata-Ostrowski) *Let two nondecreasing ordered sequences of real numbers* $\{\mu_j\}$ *and* $\{m_j\}$, $j = 0, \ldots, n-1$, *satisfy*

$$\sum_{j=0}^{k-1} \mu_j \leq \sum_{j=0}^{k-1} m_j \tag{3.7}$$

for each k. *Then for any differentiable convex function* $\Psi(x)$,

$$\sum_{j=0}^{k-1} \Psi(\mu_j) \geq \sum_{j=0}^{k-1} \Psi(m_j) + \Psi'(m_{k-1}) \cdot \sum_{j=0}^{k-1} (\mu_j - m_j).$$

In particular, assuming either that Ψ *is nonincreasing or that* $\sum_{j=0}^{k-1} \mu_j = \sum_{j=0}^{k-1} m_j$,

$$\sum_{j=0}^{k-1} \Psi(\mu_j) \geq \sum_{j=0}^{k-1} \Psi(m_j)$$

Sums of eigenvalues

- “Semiclassical” asymptotics and inequalities relate sums of eigenvalues of Laplace or Schrödinger operators to geometric properties of the spaces on which they act.

Spectral averages, geometry, and dimensionality

For Laplacians (DBC):

- Weyl law: $$\lambda_k \sim 4\pi^2 \left(\frac{k}{C_d|\Omega|}\right)^{2/d}$$
- However, averaging helps:

Spectral averages, geometry, and dimensionality

- Weyl law: $\lambda_k \sim 4\pi^2(k/C_d|\Omega|)^{2/d}$.
- Berezin-Li-Yau
$$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}$$
- Lieb Thirring for negative eigenvalues of Schrödinger operators,
$$\sum_{\lambda_j<0} |\lambda_j|^p \leq L_{p,d} \int |V(x)|^{p+d/2}$$

Variational bounds on sums

- In 1992 Pawel Kröger found a variational argument for the Neumann counterpart to Berezin-Li-Yau, i.e. a Weyl-sharp upper bounds on sums of the eigenvalues of the Neumann Laplacian.
- BLY: $$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}$$
- Kröger: $$\sum_{j=0}^{k-1} \mu_j \leq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}$$

Spectral dimension

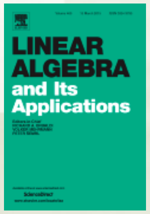
- We can use the optimal exponent in a BLY or Kröger-type bound to define the *spectral dimension*.
- Dimension in the ordinary sense is a *measure of complexity*.
- How closely can we tie the spectral dimension to a geometric dimension?

More vs less efficient embeddings of a 1D graph.

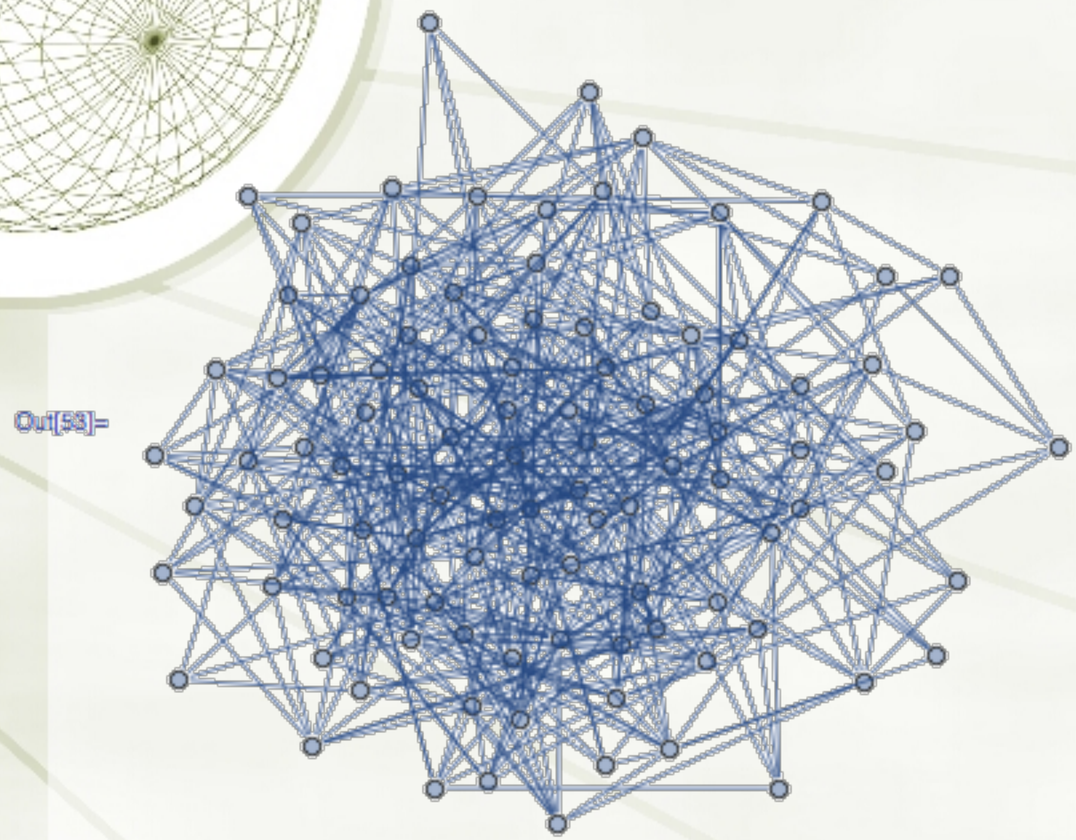
Combinatorial graphs

- A graph connects n vertices with edges as specified by an adjacency matrix A, with a_{ij} = 1 when i and j are connected, otherwise 0. The graph is not a priori living in Euclidean space.
- There is a natural Laplacian on the graph, with non-negative spectrum, which has been heavily studied, but not much with this question in mind.

Harrell-Stubbe LAA, 2014



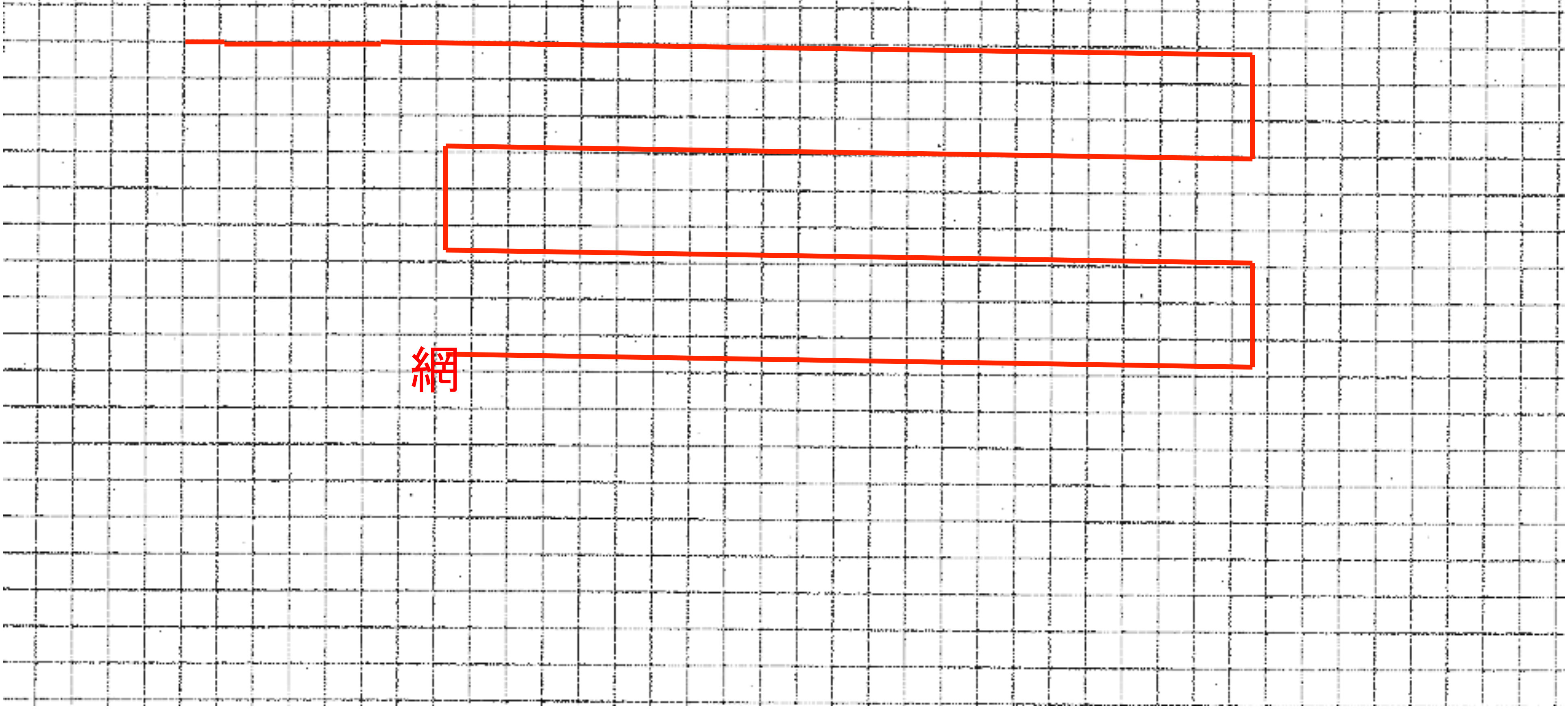
Dimension and complexity



This is a randomly generated “graph” showing 520 connections among 100 items. How many independent kinds of information (“dimensions”) are there?

Another interesting question:

Can you distinguish dimensions on different scales?



A new tool: an averaged variational principle for sums

Proof. By integrating (3.1),

$$\mu_k \int_{\mathfrak{M}_0} \left(\langle f_\zeta, f_\zeta\rangle - \langle P_{k-1} f, P_{k-1} f_\zeta\rangle\right) d\sigma \tag{3.3}$$
$$\leq \int_{\mathfrak{M}_0} \langle M f_\zeta, f_\zeta\rangle \, d\sigma - \int_{\mathfrak{M}_0} \langle M P_{k-1} f_\zeta, P_{k-1} f_\zeta\rangle \, d\sigma,$$

or

$$\mu_k \int_{\mathfrak{M}_0} \left(\langle f_\zeta, f_\zeta\rangle - \sum_{j=0}^{k-1} |\langle f_\zeta, \psi^{(j)}\rangle|^2\right) d\sigma \tag{3.4}$$
$$\leq \int_{\mathfrak{M}_0} \langle M f_\zeta, f_\zeta\rangle \, d\sigma - \int_{\mathfrak{M}_0} \sum_{j=0}^{k-1} \mu_j |\langle f_\zeta, \psi^{(j)}\rangle|^2 \, d\sigma.$$

Since μ_k is larger than or equal to any weighted average of $\mu_1 \dots \mu_{k-1}$, we add to (3.4) the inequality

$$-\mu_k \int_{\mathfrak{M}\setminus\mathfrak{M}_0} \left(\sum_{j=0}^{k-1} |\langle f_\zeta, \psi^{(j)}\rangle|^2\right) d\sigma \leq - \int_{\mathfrak{M}\setminus\mathfrak{M}_0} \sum_{j=0}^{k-1} \mu_j |\langle f_\zeta, \psi^{(j)}\rangle|^2 \, d\sigma, \tag{3.5}$$

and obtain the claim. □

How to use the averaged variational principle to get sharp results?

Ans: If $\mathfrak{M}_0$ is large enough that

$$\int_{\mathfrak{M}_0} \langle f_\zeta, f_\zeta \rangle \, d\sigma \geq \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma$$

then

$$\sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma \leq \int_{\mathfrak{M}_0} \langle M f_\zeta, f_\zeta \rangle d\sigma$$

Recent applications of the averaged variational principle:

1. Harrell-Stubbe, LAA 2014: Weyl-type upper bounds on sums of eigenvalues of (discrete) graph Laplacians and related operators.
2. El Soufi-Harrell-Ilias-Stubbe, nearing preprint stage: Semiclassically sharp Kröger-type results for a large family of 2nd order PDEs on manifolds.
3. Harrell-Dever, stuff on blackboards: Quantum graphs.

Example: Recover Kröger's result

With the Parseval identity,

$$\int_{\mathfrak{M}} |\left\langle e^{i\mathbf{p}\cdot\mathbf{x}}, \psi^{(j)} \right\rangle|^2 = (2\pi)^d \|\psi^{(j)}\|^2 = (2\pi)^d.$$

IF $|\mathfrak{M}_0||\Omega| \geq (2\pi)^d k,$ then

$$(2\pi)^d \sum_{j=0}^{k-1} \mu_j \leq \int_{\mathfrak{M}_0} |\mathbf{p}|^2 |\Omega|$$

Choosing $\mathfrak{M}_0$ as a ball of radius R in p-space, a simple calculation gives Kröger.

Bounds on sums for quantum graphs

Let us assume that our quantum graph consists of a finite number of straight lines, which can be isometrically embedded in d-dimensional Euclidean space. We'll define the Hamiltonian as the Friedrichs extension of the quadratic form

$$Q(\varphi, \varphi) = \sum_{\mathcal{E} \subset \Gamma} \int_{\mathcal{E}} \left(|\varphi'|^2 + V(x)|\varphi|^2 \right) dx$$

on functions $\varphi \in H^1(\Gamma)$, interpreted as the orthogonal sum of H^1 on the edges, with Lebesgue measure.

For today we'll set V=0, and avoid the temptation to introduce other complications.
Well, other than some general remarks.

Bounds on sums for quantum graphs

To get the machine running, we'd like a set of trial functions which have a nice relation to the operator and a completeness relation, so the Fourier exponentials again come to mind.

An adapted Fourier transform

If $f(x) \in L^2(\Gamma)$, we can define a Fourier transform adapted to the graph by

$$\hat{f}(\mathbf{p}) := \frac{1}{\sqrt{2\pi}} \int_\Gamma f(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{p}}$$

Inverse transform:

$$\check{g}(\mathbf{x}) = \mathfrak{F}^{-1}[g]\,\mathbf{x} := \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \frac{1}{(2\pi n)^{d-1}} \int_{-\pi n}^{\pi n} \cdots \int_{-\pi n}^{\pi n} g(\mathbf{p}) e^{i\mathbf{x}\cdot\mathbf{p}} d^d p$$

Parseval relation:

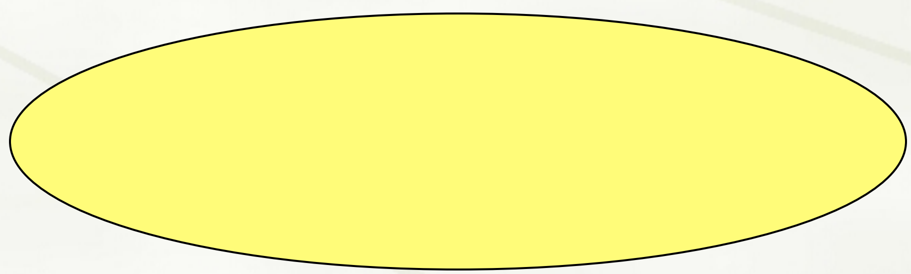
$$\langle f, g\rangle = \left\langle \hat{f}, \hat{g} \right\rangle_* = \lim_{n\to\infty} \frac{1}{(2\pi n)^{d-1}} \int_{-\pi n}^{\pi n} \cdots \int_{-\pi n}^{\pi n} \hat{f}\overline{\hat{g}} d^d p$$

Bounds on sums for quantum graphs

So, what happens when you use $f_\zeta = \frac{1}{\sqrt{2\pi}} e^{-i\mathbf{p}\cdot\mathbf{x}}$ as a test function in the AVP?

$$Q(f_\zeta, f_\zeta) = \sum_{\mathcal{E}\subset\Gamma} \int_{\mathcal{E}} |p_{\mathcal{E}}|^2 =: q_\Gamma(\mathbf{p}),$$

which defines a certain *phase-space ellipsoid* via $q_\Gamma^{-1}(1)$.



Bounds on sums for quantum graphs

Now, if q is homogeneous function of degree h, with a scaling argument, we find that

$$\int_{q(\mathbf{p})\leq\Lambda} q(\mathbf{p}) d^d p = \frac{d}{d+h}(V(1))^{-d/h}\,(V(\Lambda))^{1+\frac{h}{d}}\,,$$

where $V(\Lambda)$ designates the volume of the phase-space region with maximum energy Λ. (Here $h = 2$, but other powers are sometimes of interest.) By the theorem, we need to set $V(\Lambda) = \frac{k}{|\Gamma|}$.

Conclusion. For quantum graphs,

$$\sum_{\ell=0}^{k} \mu_k \leq \frac{d}{d+2}(V(1))^{-d/2}\left(\frac{k}{|\Gamma|}\right)^{1+\frac{2}{d}}$$

Bounds on sums for quantum graphs

$$m_k(1-\sqrt{1-S_k}) \le \mu_{k+1} \le m_k(1+\sqrt{1-S_k})$$

$$S_k = \frac{\frac{d+2}{d}\frac{1}{k}\sum_{i=1}^{k}\mu_i}{m_k}.$$

This result is in the form that applies to the case of Euclidean domains, where m_k is the Weyl expression, but a similar result works for all of our applications, including quantum graphs. (Harrell-Stubbe, unpublished)

The mother of all upper bounds on sums for PDEs

We (El Soufi, Harrell, Ilias, Stubbe) recently used the A.V.P. to get upper bounds for sums of eigenvalues of corresponding to quadratic forms.

$$\mathcal{E}(\varphi) := \frac{\int_\Omega \left(|\nabla\varphi(\mathbf{x})|^2 + V(\mathbf{x})|\varphi(\mathbf{x})|^2\right) w(\mathbf{x}) e^{-2\rho(\mathbf{x})} dv_g}{\int_\Omega |\varphi(\mathbf{x})|^2 e^{-2\rho(\mathbf{x})} dv_g},$$

where Ω is a domain in a general Riemannian manifold.

An adapted Fourier transform

Let $F : (M, g) \to \mathbb{R}^N$, be an isometric embedding (whose existence for sufficiently large N is guaranteed by Nash's embedding Theorem). To any function $u \in L^2(\Omega)$, we associate the function $\hat{u}_F : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\hat{u}_F(\mathbf{p}) = \int_\Omega u(\mathbf{x}) e^{i\mathbf{p}\cdot F(\mathbf{x})} dv_g$$

(see [1, Theorem 2.1] [10, Theorem 7.1.26], [16, Corollary 5.2]), that there exists a constant $C_{F(\Omega)}$ such that, $\forall u \in L^2(\Omega)$ and $\forall R > 0$,

$$\int_{B_R} |\hat{u}_F(\mathbf{p})|^2 d^N p \leq C_{F(\Omega)} R^{N-\nu} \|u\|^2 \tag{17}$$

where B_R is the Euclidean ball of radius R in $\mathbb{R}^N$ centered at the origin and $\|u\|^2 = \int_\Omega u^2 dv_g$.

We define the Riemannian constant H_Ω by

$$H_\Omega = \inf_{N \geq \nu} \inf_{F \in I(M, \mathbb{R}^N)} \left(\frac{\nu + 2}{N + 2} \right)^{\frac{\nu}{2}} \frac{1}{\omega_N} C_{F(\Omega)} \tag{18}$$

If the manifold is conformal to a homogeneous spaces, more precise bounds are obtained

A homogeneous space is a manifold M with a continuous symmetry group of isomorphisms M→M.

Canonical examples: R^d, S^d, H^d.

Corollary 3.4. *Let Ω be a bounded domain of $\mathbb{R}^\nu$ and let $g = \alpha^2 g_E$ be a Riemannian metric that is conformal to the Euclidean metric g_E. The Neumann eigenvalues $\{\mu_l\}$ of the Laplacian Δ_g in Ω satisfy the following estimates in which $|\Omega|$ denotes the Euclidean volume of Ω :*

(1) For all $z \in \mathbb{R}$,

$$\sum_{j\geq 0}(z-\mu_j)_+ \geq \frac{2\omega_\nu|\Omega|}{(\nu+2)(2\pi)^\nu}\left(\fint_\Omega \alpha^{-2}\, d^\nu x\right)^{\frac{\nu}{2}}\left(z-\frac{\nu^2}{4}\fint_\Omega |\nabla\alpha|^2\alpha^{-4}\, d^\nu x\right)_+^{1+\frac{\nu}{2}} \tag{44}$$

(2) For all $k \in \mathbb{N}$,

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \leq \frac{\nu}{\nu+2}4\pi^2\left(\frac{k}{\omega_\nu|\Omega|}\right)^{\frac{2}{\nu}}\fint_\Omega \alpha^{-2}\, d^\nu x + \frac{\nu^2}{4}\fint_\Omega |\nabla\alpha|^2\alpha^{-4}\, d^\nu x. \tag{45}$$

(3) For all $k \in \mathbb{N}$,

$$\mu_k\left(1-\frac{\nu^2}{4}\frac{\fint_\Omega |\nabla\alpha|^2\alpha^{-4}\, d^\nu x}{\mu_k}\right)_+^{1+\frac{2}{\nu}} \leq 4\left(\frac{\nu+2}{2}\right)^{\frac{2}{\nu}}\pi^2\left(\frac{k}{\omega_\nu|\Omega|}\right)^{\frac{2}{\nu}}\fint_\Omega \alpha^{-2}\, d^\nu x. \tag{46}$$

In particular,

$$\mu_k \leq \max\left\{\frac{\nu^2}{2}\fint_\Omega |\nabla\alpha|^2\alpha^{-4}\, d^\nu x\ ;\ 8\,(\nu+2)^{\frac{2}{\nu}}\,\pi^2\left(\frac{k}{\omega_\nu|\Omega|}\right)^{\frac{2}{\nu}}\fint_\Omega \alpha^{-2}\, d^\nu x\right\}. \tag{47}$$

For example, a domain of the hyperbolic space $\mathbf{H}^\nu$ can be identified with a domain of the Euclidean unit ball endowed with the metric $g = \left(\frac{2}{1-|x|^2}\right)^2 g_E$. Corollary 3.4 gives for such a domain, with $\alpha = \frac{2}{1-|x|^2}$, $\fint_\Omega \alpha^{-2}\, d^\nu x \leq \frac{1}{4}$ and $\fint_\Omega |\nabla\alpha|^2\alpha^{-4}\, d^\nu x = \fint_\Omega |x|^2\, d^\nu x$,

Coherent states

For domains conformal to Euclidean sets, we take

$$f_\varsigma(\mathbf{x}) := \frac{1}{(2\pi)^{\nu/2}} e^{i\mathbf{p}\cdot(\mathbf{x})+\rho(\mathbf{x})} h(\mathbf{x}-\mathbf{y}).$$

and reason as follows

Some definitions

- The L^2-normalized ground-state Dirichlet eigenfunction for the ball of geodesic radius r in M will be denoted h_r and $\mathcal{K}(h_r) := \int_{B_r} |\nabla h_r(\mathbf{x})|^2 d^\nu x$. I.e., in this section where $M = \mathbb{R}^\nu$, h is a scaled Bessel function and

$$\mathcal{K}(h_r) = \frac{j^2_{\frac{\nu}{2}-1,1}}{r^2}.$$

- $L(\Lambda)$ will denote the maximal Lipschitz constant of $\widetilde{V}(\mathbf{x})$ on the region $\Omega \cap \{\mathbf{x} : \widetilde{V}(\mathbf{x}) \leq \Lambda\}$.

Definition 4.2. *The* Euclidean phase-space volume *for energy* Λ *is defined as*

$$\Phi_1(\Lambda) := \frac{1}{(2\pi)^\nu}\left|(\mathbf{x},\mathbf{p}) : |\mathbf{p}|^2 + \widetilde{V}(\mathbf{x}) \leq \Lambda\right| = \frac{\omega_\nu}{(2\pi)^\nu}\int_\Omega \left(\Lambda - \widetilde{V}(\mathbf{x})\right)_+^{\frac{\nu}{2}} d^\nu x,$$

according to a standard calculation to be found, for example, in [13]. *Here* ω_ν *is the volume of the unit ball in dimension* ν *and* $(x)_+ := max(x, 0)$. *If the weight in* (2) *is not constant, we make use of a* weighted phase-space volume,

$$\Phi_w(\Lambda) = \frac{\omega_\nu}{(2\pi)^\nu}\int_\Omega \left(\Lambda - \widetilde{V}(\mathbf{x})\right)_+^{\frac{\nu}{2}} w(\mathbf{x}) d^\nu x.$$

The total energy associated with this quantity is correspondingly

$$\begin{aligned} E_w(\Lambda) &:= \frac{1}{(2\pi)^\nu}\int_{\{(\mathbf{x},\mathbf{p}):\mathbf{x}\in\Omega,|\mathbf{p}|^2+\widetilde{V}(\mathbf{x})\leq\Lambda\}} \left(|\mathbf{p}|^2 + \widetilde{V}(\mathbf{x})\right) w(\mathbf{x}) d^\nu x d^\nu p \\ &= \frac{\nu}{\nu+2}\frac{\omega_\nu}{(2\pi)^\nu}\int_\Omega \left(\Lambda - \widetilde{V}(\mathbf{x})\right)_+^{1+\frac{\nu}{2}} w(\mathbf{x}) d^\nu x. \qquad (51) \end{aligned}$$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