Eleven spectral properties of the sphere

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From Carthage to the World

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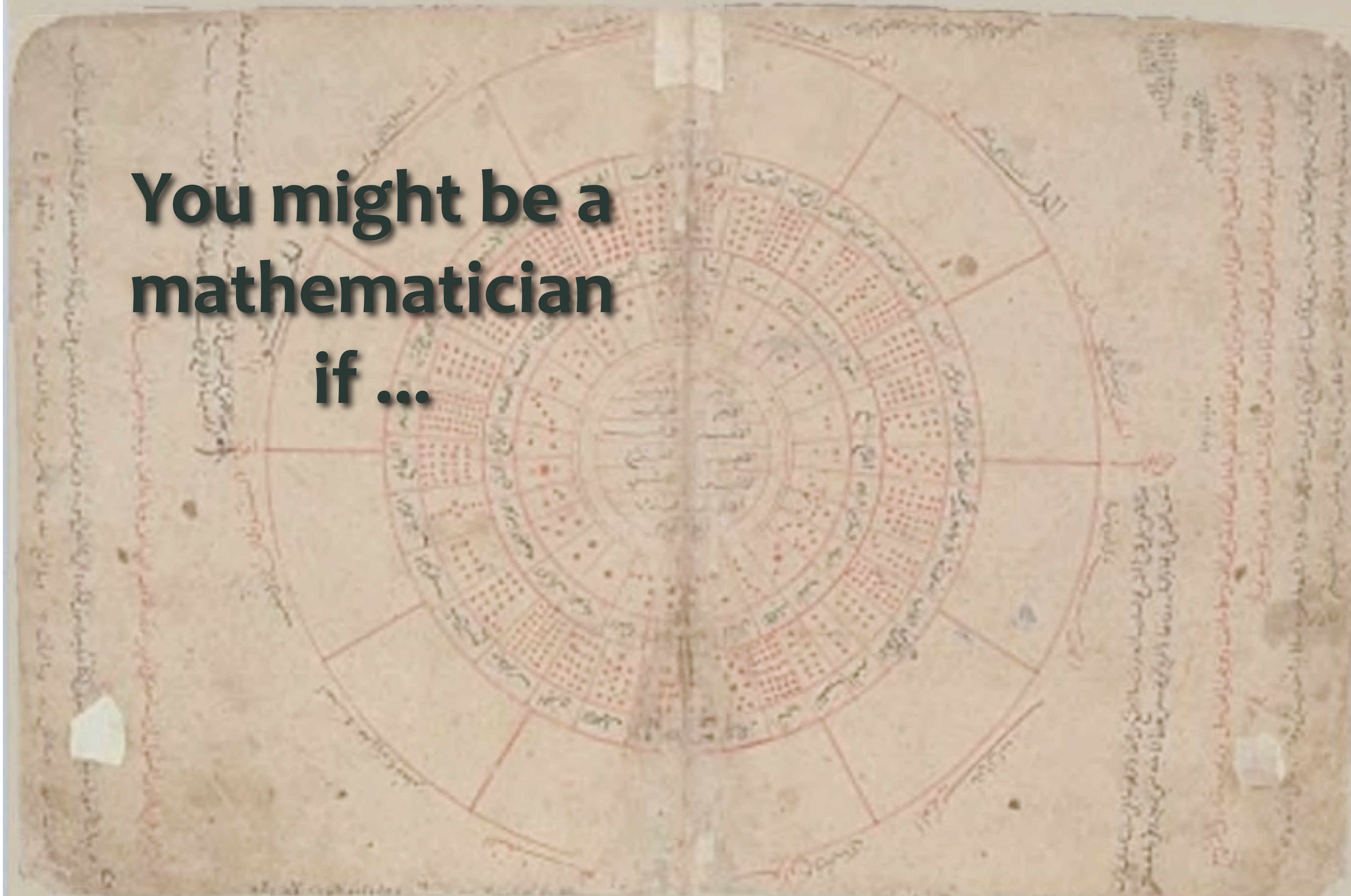
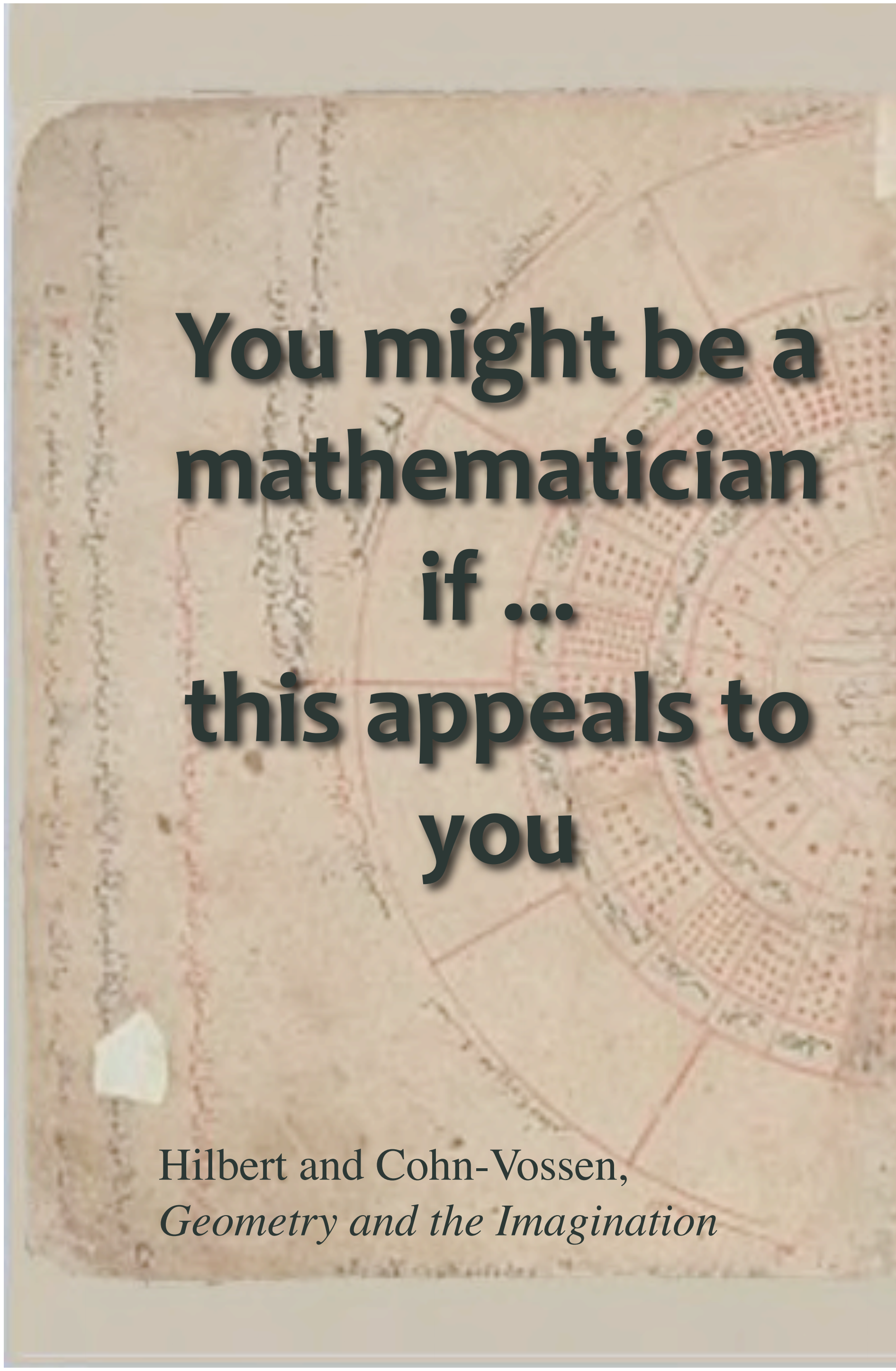
You might be a mathematician if …



Background picture "Diagram of the Universe" (c) Bodleian Library, University of Oxford MS Arab c.90 folio #2b-3a



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Hilbert and Cohn-Vossen,
Geometry and the Imagination

§ 32. Eleven Properties of the Sphere

We have already become acquainted with the surfaces of vanishing Gaussian curvature. We shall now look for the surfaces of constant positive or negative curvature. By far the simplest and most important surface of this type is the sphere. A thorough study of the sphere would in itself provide sufficient material for a whole book. We shall here present only eleven properties that have a particularly strong appeal to the visual intuition. We shall at the same time become acquainted with several properties that are of importance not only for the geometry of the sphere but also for the general theory of surfaces. With regard to each property to be described we shall inquire whether it defines the sphere uniquely or whether there are other surfaces having the given property.

1. *The points of a sphere are equidistant from a fixed point. Also, the ratio of the distances of its points from two fixed points is constant.*

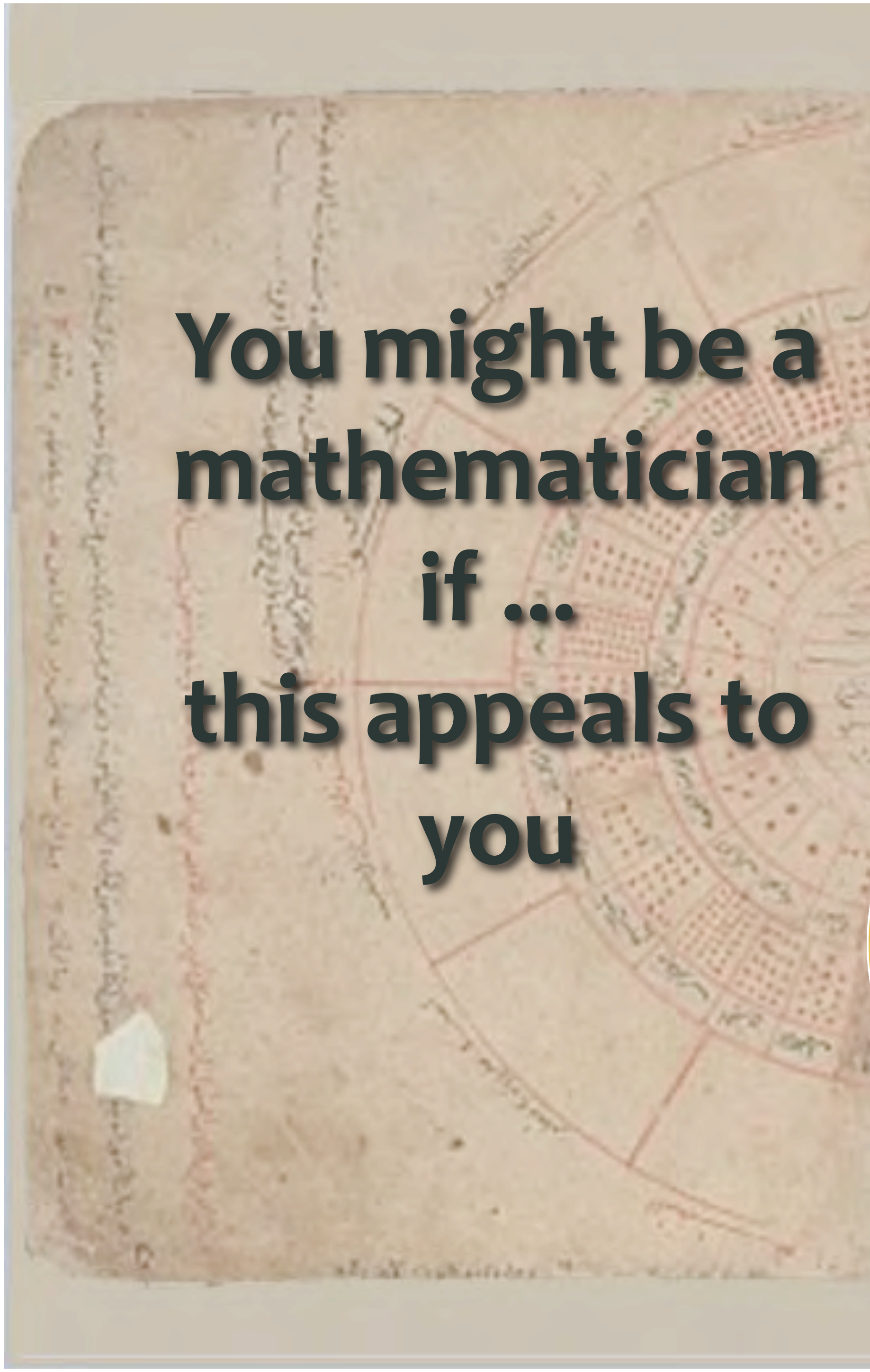
The first of these two properties constitutes the elementary definition of the sphere and consequently defines the sphere uniquely. The fact that the sphere has the second property as well, can be ascertained very easily by analytical methods. On the other hand, the second property defines not only the sphere but the plane as well. For, a plane is obtained if, and only if, the constant ratio is equal to unity. The plane obtained in this case is the plane of symmetry of the two fixed points.

2. *The contours and the plane sections of the sphere are circles.*

In the discussion of the second-order surfaces we mentioned the theorem that all the plane sections and contours of such surfaces are conics. In the case of a sphere, all these conics are circles. This property defines the sphere uniquely. From the observation that the shadow of the earth at a lunar eclipse is always a circle we may therefore infer that the earth is spherical.

3. *The sphere has constant width and constant girth.*

The term *constant width* denotes the property, of a solid, that the distance between any pair of parallel tangent planes is constant. Thus a sphere can be rolled arbitrarily between two parallel tangent planes. It would seem plausible that the sphere is uniquely defined by this property. In actual fact, however, there are numerous other closed convex surfaces, some of them without any singularities, whose



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sheets of the confocal system defined by E. In this process, it is necessary to include the focal hyperbola as a limiting case of a hyperboloid and to count all the straight lines meeting the hyperbola as tangents to this degenerate surface. The focal hyperbola intersects E in the four umbilical points. A limiting process applied to the above argument shows that the family of geodesic lines of E belonging to the focal hyperbola consists of all those geodesics that pass through an umbilical point of E, and only of those.[5] Furthermore, it is found that every geodesic line through an umbilical point also passes through the diametrically opposite umbilical point.

On the sphere, all the geodesics through a given point P also pass through a second fixed point, the point diametrically opposite P. The behavior of the geodesic lines passing through an umbilical point of the ellipsoid is analogous to this property. On the other hand, it can be proved that the geodesics through any other fixed point of the ellipsoid do not all have a second point in common.

It is natural to ask whether the sphere is the only surface on which all the geodesic lines emanating from an arbitrary fixed point have a second point in common. The answer to this question has not yet been found.

7. *Of all solids having a given volume, the sphere is the one having the smallest surface area; of all solids having a given surface area, the sphere is the one having the greatest volume.*

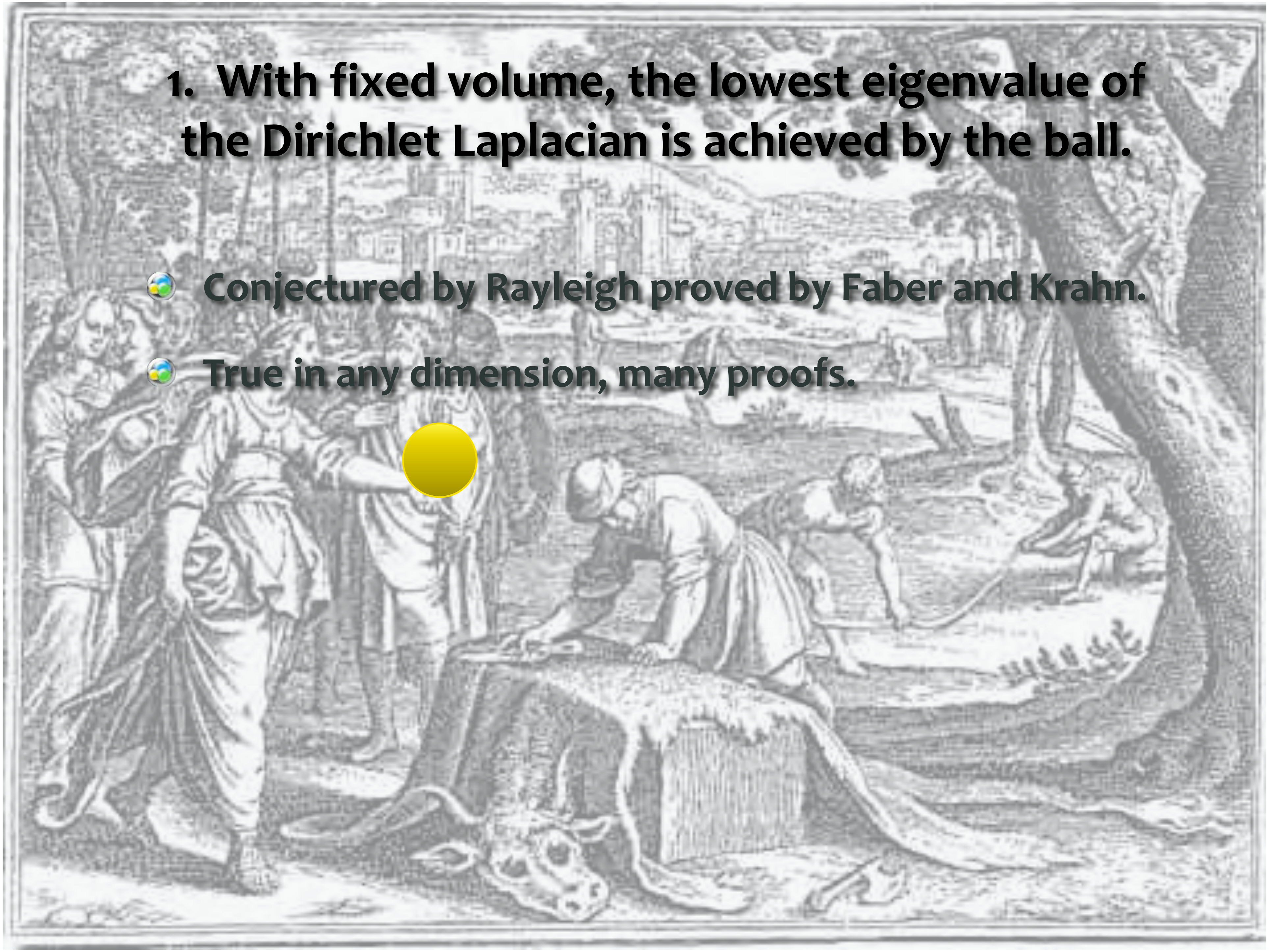
These two properties (each of which implies the other) define the sphere uniquely. The proof of this fact leads to a problem of the calculus of variations and is extremely laborious. But a simple experimental proof is implicit in every freely floating soap bubble. As was mentioned earlier in connection with the minimal surfaces, the soap bubble, by virtue of its surface tension, seeks to reduce its surface area to a minimum; and since the bubble encloses a fixed volume of air, it follows that the bubble assumes the minimum surface area for a fixed volume. But it is found by observation that freely floating soap bubbles are always spherical unless they are appreciably subjected to the influence of gravity because of adhering drops of liquid.

8. *The sphere has the smallest total mean curvature among all convex solids with a given surface area.*

[5] The thread construction described on p. 188 is intimately connected with this fact.

1. With fixed volume, the lowest eigenvalue of the Dirichlet Laplacian is achieved by the ball.

- Conjectured by Rayleigh proved by Faber and Krahn.
- True in any dimension, many proofs.

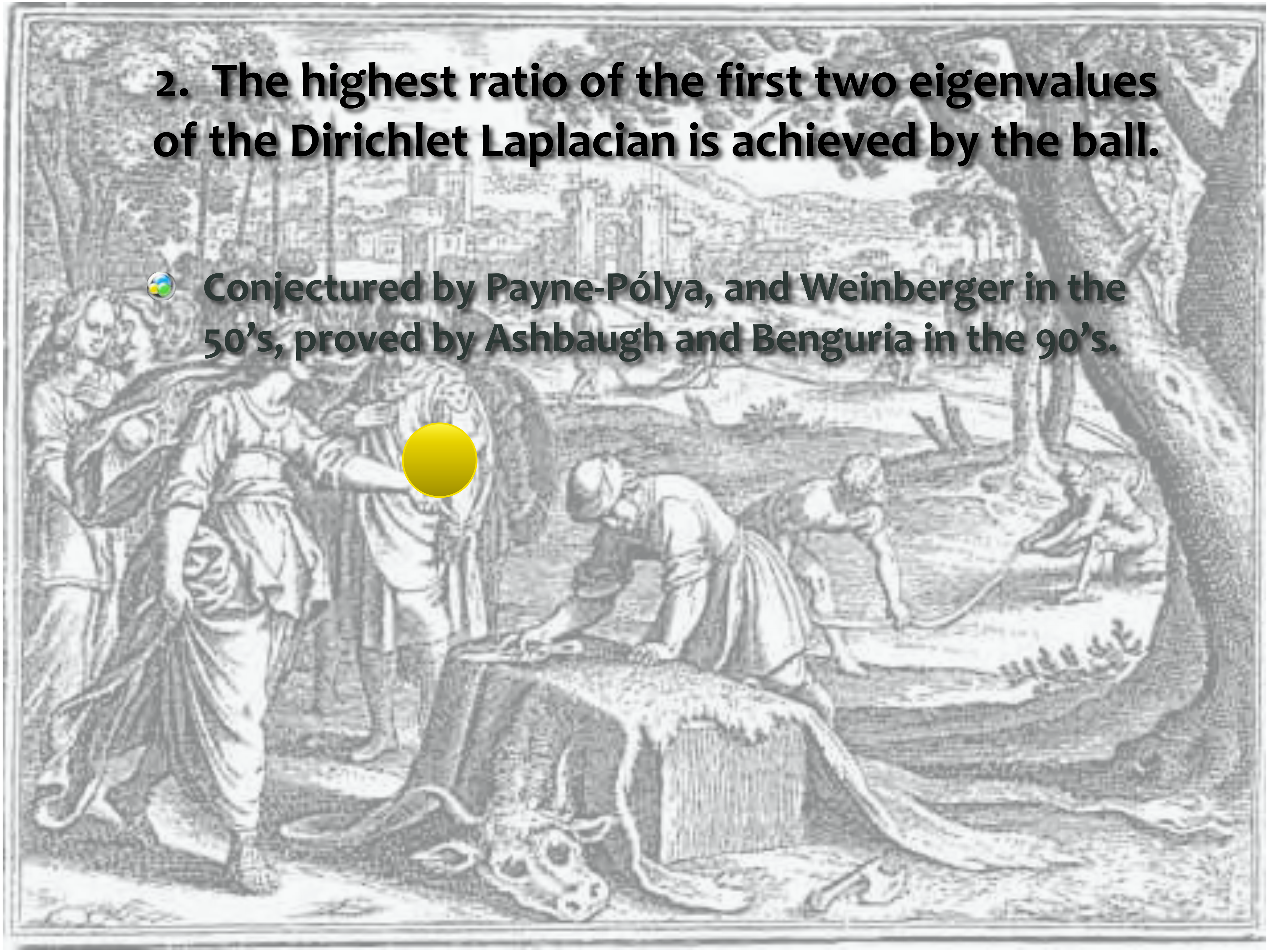


1. **With fixed volume, the lowest eigenvalue of the Dirichlet Laplacian is achieved by the ball.**

- **Clever proof of Chavel: with the co-area formula, the essential step is in fact the original isoperimetric theorem for volume and surface.**

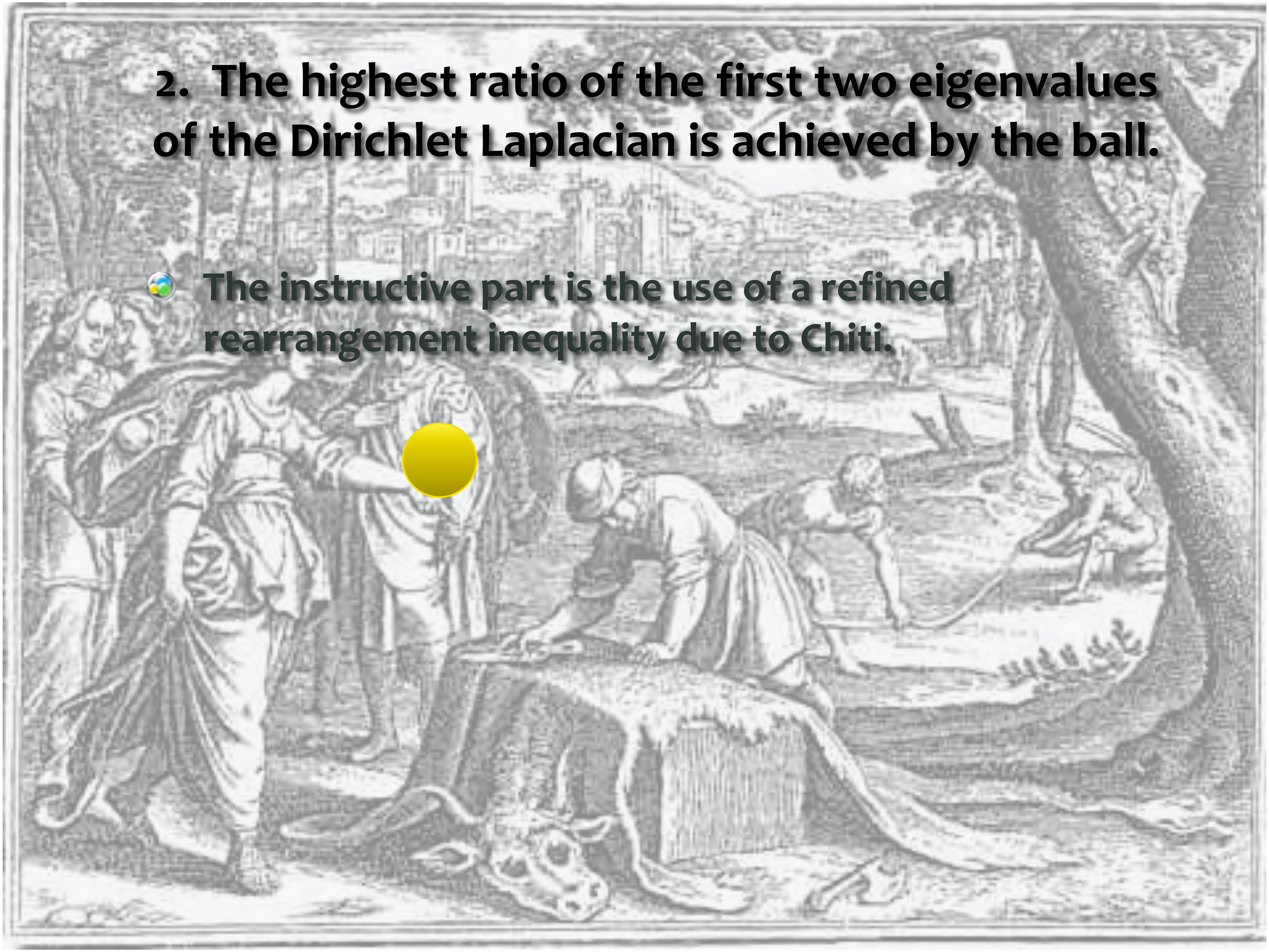
2. The highest ratio of the first two eigenvalues of the Dirichlet Laplacian is achieved by the ball.

- Conjectured by Payne-Pólya, and Weinberger in the 50's, proved by Ashbaugh and Benguria in the 90's.



2. The highest ratio of the first two eigenvalues of the Dirichlet Laplacian is achieved by the ball.

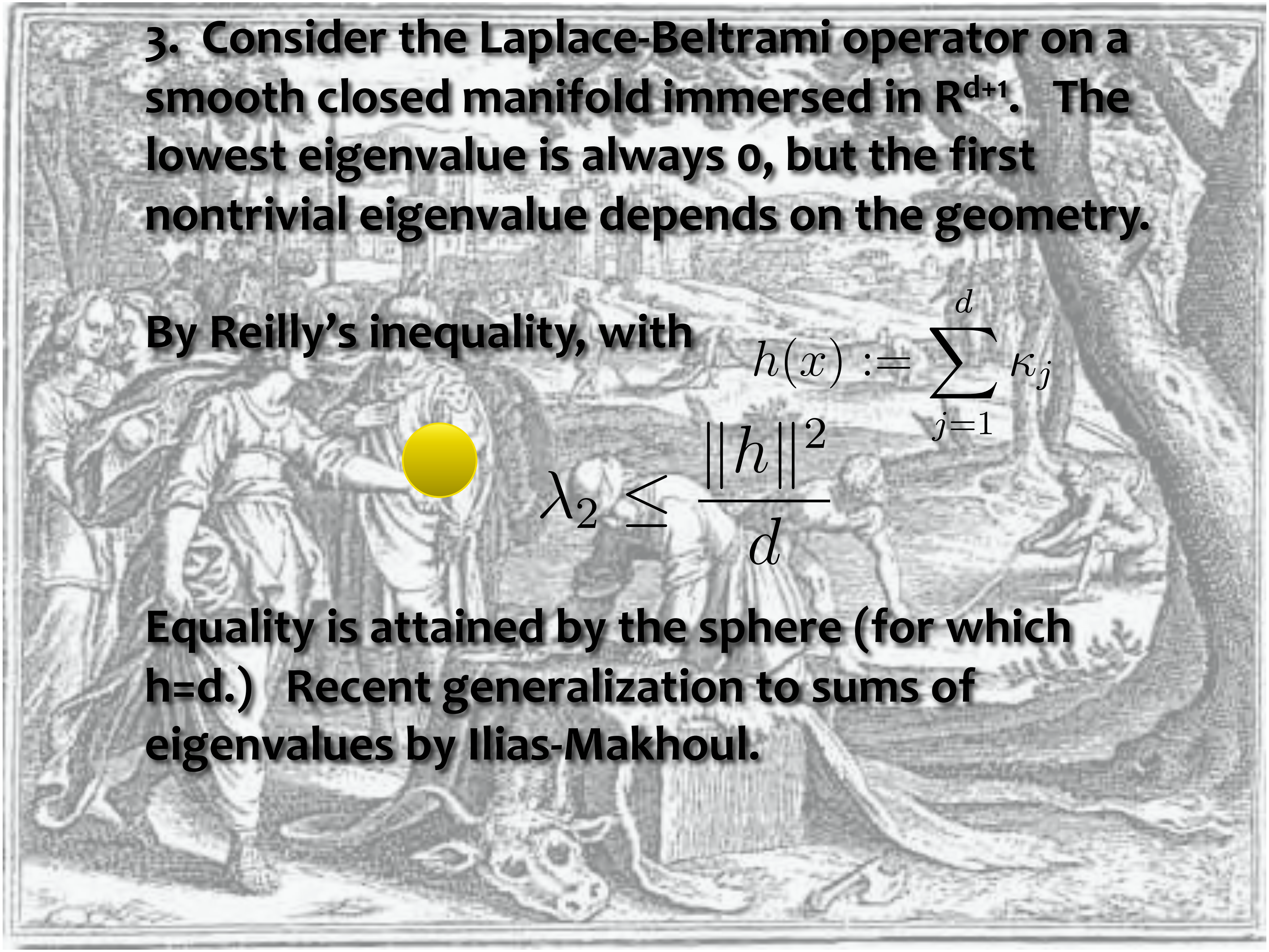
- The instructive part is the use of a refined rearrangement inequality due to Chiti.



3. Consider the Laplace-Beltrami operator on a smooth closed manifold immersed in R^{d+1}. The lowest eigenvalue is always 0, but the first nontrivial eigenvalue depends on the geometry.

According to Reilly's inequality,

$$\lambda_2 \leq \frac{\|h\|^2}{d}$$



3. Consider the Laplace-Beltrami operator on a smooth closed manifold immersed in R^{d+1}. The lowest eigenvalue is always 0, but the first nontrivial eigenvalue depends on the geometry.

By Reilly's inequality, with

$$h(x) := \sum_{j=1}^{d} \kappa_j$$

$$\lambda_2 \leq \frac{||h||^2}{d}$$

Equality is attained by the sphere (for which h=d.) Recent generalization to sums of eigenvalues by Ilias-Makhoul.

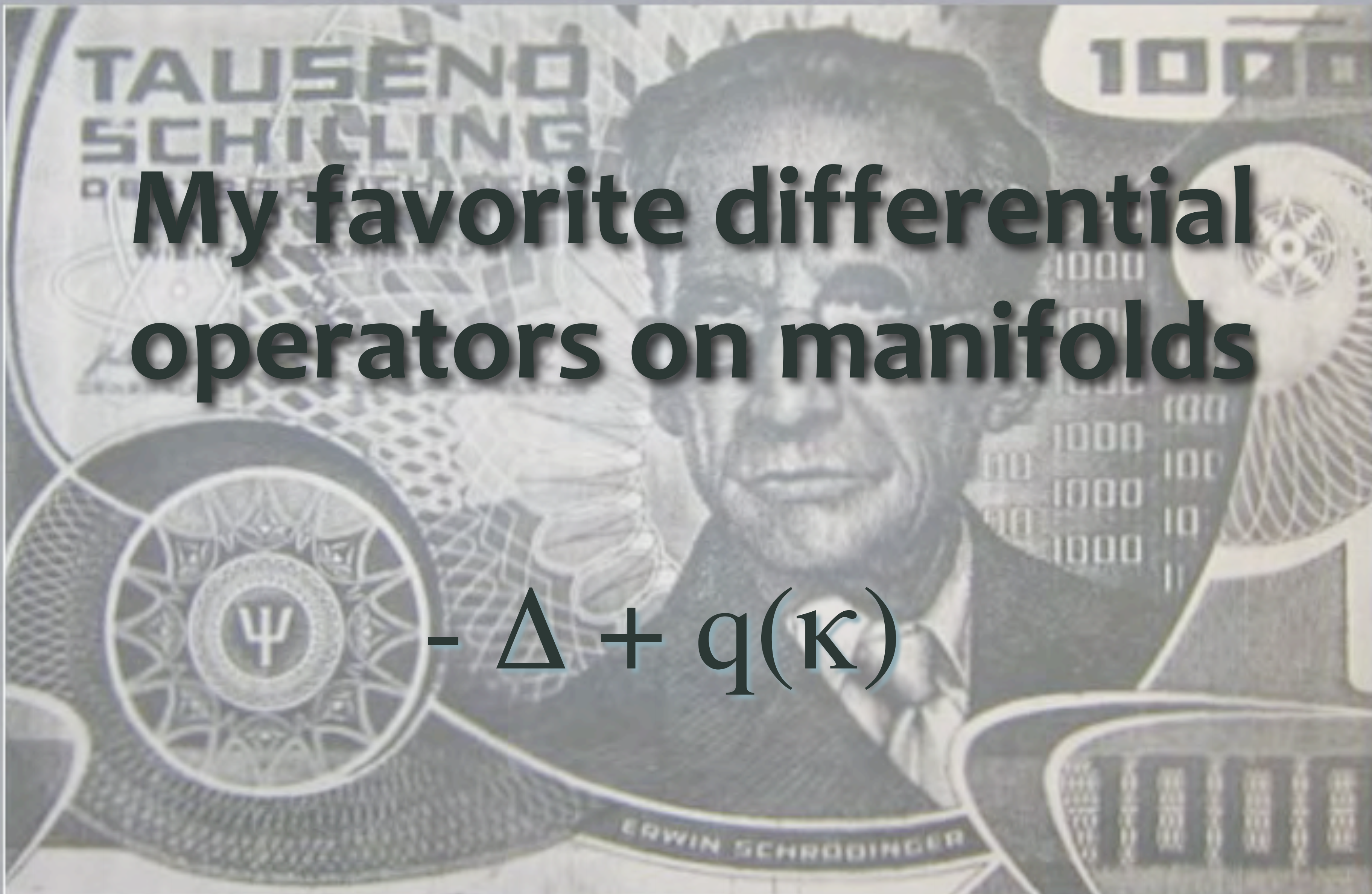
4. Now consider the Laplace-Beltrami operator on a closed manifold in 2D, of fixed surface area, and of the topological type of the sphere. The first positive eigenvalue is maximizedby the sphere. (J. Hersch, CRAS 1970)

4. Now consider the Laplace-Beltrami operator on a closed manifold in 2D, of fixed surface area, and of the topological type of the sphere. The first positive eigenvalue is maximized by the sphere. (J. Hersch, CRAS 1970)

- The surface can be mapped conformally to the unit sphere in R^3. The preimages of the Cartesian coordinates in R^3 are great trial functions

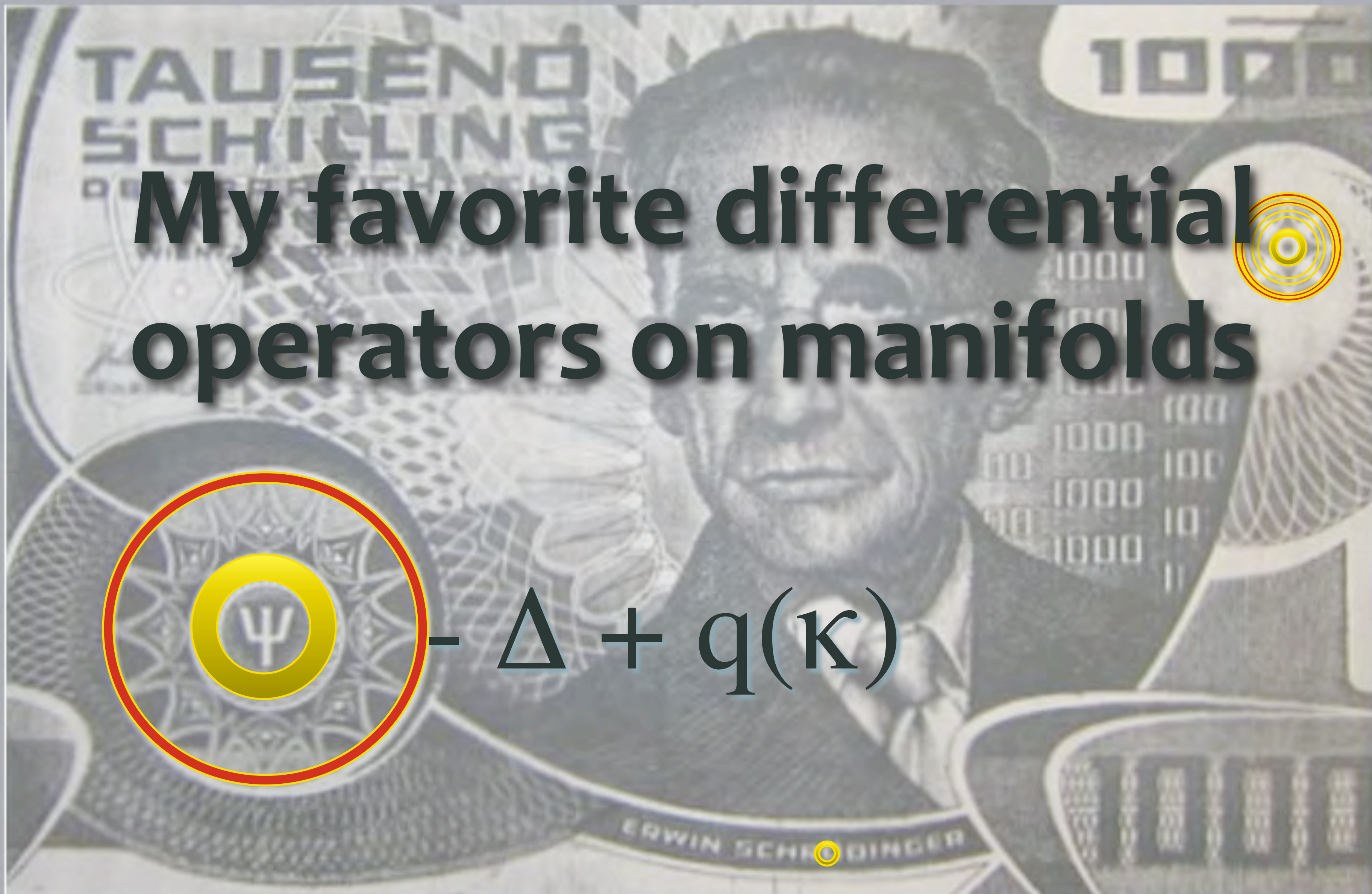
My favorite differential operators on manifolds

$$-\Delta + q(\kappa)$$



My favorite differential operators on manifolds

Ψ $- \Delta + q(\kappa)$



5. Subtract Gauss curvature from the Laplace-Beltrami operator on a closed manifold in 2D. The second eigenvalue is still maximized by the sphere (Harrell, JDGA 1996)**.**

$$-\Delta + q(\kappa),$$

$$q(\kappa) = - g\, \kappa_1 \kappa_2$$

The conformal equivalence of M and S^2.

Lemma. *(J. Hersch). Let* Ω *be a two-dimensional, closed, smooth Riemannian manifold of the topological type of the sphere, and specify a bounded, positive, measurable function* ρ *on* Ω*. Then there exists a conformal transformation* $\Phi : \Omega \to S^2 \subset R^3$*, embedded in the standard way as the unit sphere, such that*

$$\int_{S^2} \mathbf{x}\rho(\Phi^{-1}(\mathbf{x}))J d\hat{S} = \mathbf{0}. \tag{3}$$

Jacobian

5. Subtract Gauss curvature from the Laplace-Beltrami operator on a closed manifold in 2D. The second eigenvalue is still maximized by the sphere (Harrell, JDGA 1996).

For the trial function φ let's choose one of the Cartesian coordinates x,y,z on S^2, but "pull back" to Ω with the inverse of Hersch's conformal transformation. Let the resulting functions on Ω be called X,Y,Z. What do we know about X,Y,Z?

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1. **The functions X,Y,Z are orthogonal, because the functions x,y,z are orthogonal on S^2.**

   *** Note: The restrictions of x,y,z to S^2 are the spherical harmonics = eigenfunctions:**

   $- \nabla^2 x = 2\, x,$

   $- \nabla^2 y = 2\, y,$

   $- \nabla^2 z = 2\, z,$

5. Subtract Gauss curvature from the Laplace-Beltrami operator on a closed manifold in 2D. The second eigenvalue is still maximized by the sphere.

For the trial function φ let's choose one of the Cartesian coordinates x,y,z on S^2, but "pull back" to Ω with the inverse of Hersch's conformal transformation. Let the resulting functions on Ω be called X,Y,Z. What do we know about X,Y,Z?

1. **The functions X,Y,Z are orthogonal to one another *and also to ρ = the first eigenfunction.***

2. **$X^2 + Y^2 + Z^2 = 1$, because $x^2 + y^2 + z^2 = 1$.**

5. Add a multiple of Gauss curvature from the Laplace-Beltrami operator on a closed manifold in 2D. The second eigenvalue is still maximized by the sphere.

For the trial function φ let's choose one of the Cartesian coordinates x,y,z on S^2, but "pull back" to Ω with the inverse of Hersch's conformal transformation. Let the resulting functions on Ω be called X,Y,Z. What do we know about X,Y,Z?

1. **The functions X,Y,Z are orthogonal.**

2. **$X^2 + Y^2 + Z^2 = 1$, because $x^2 + y^2 + z^2 = 1$.**

3. **Identifying now ρ with u_1,**

$\langle X,u_1 \rangle$ = $\int_{S^2} \mathbf{x}\rho(\Phi^{-1}(\mathbf{x}))Jd\hat{S} = \mathbf{0}.$ **Likewise for Y, Z.**

Ready to roll with Rayleigh and Ritz:

Let's choose the trial function in

$$R(\zeta) := \frac{\int_\Omega |\nabla \zeta|^2 dS - g \int_\Omega \kappa_1 \kappa_2 |\zeta|^2 dS}{\int_\Omega |\zeta|^2 dS}$$

as ζ =X, Y, or Z. Considering for example X, conformality implies that

$$\int_\Omega |\nabla X|^2 dS = \int_{S^2} |\nabla x|^2 d\hat{S} = \frac{8\pi}{3}$$

Ready to roll with Rayleigh and Ritz:

Observing that

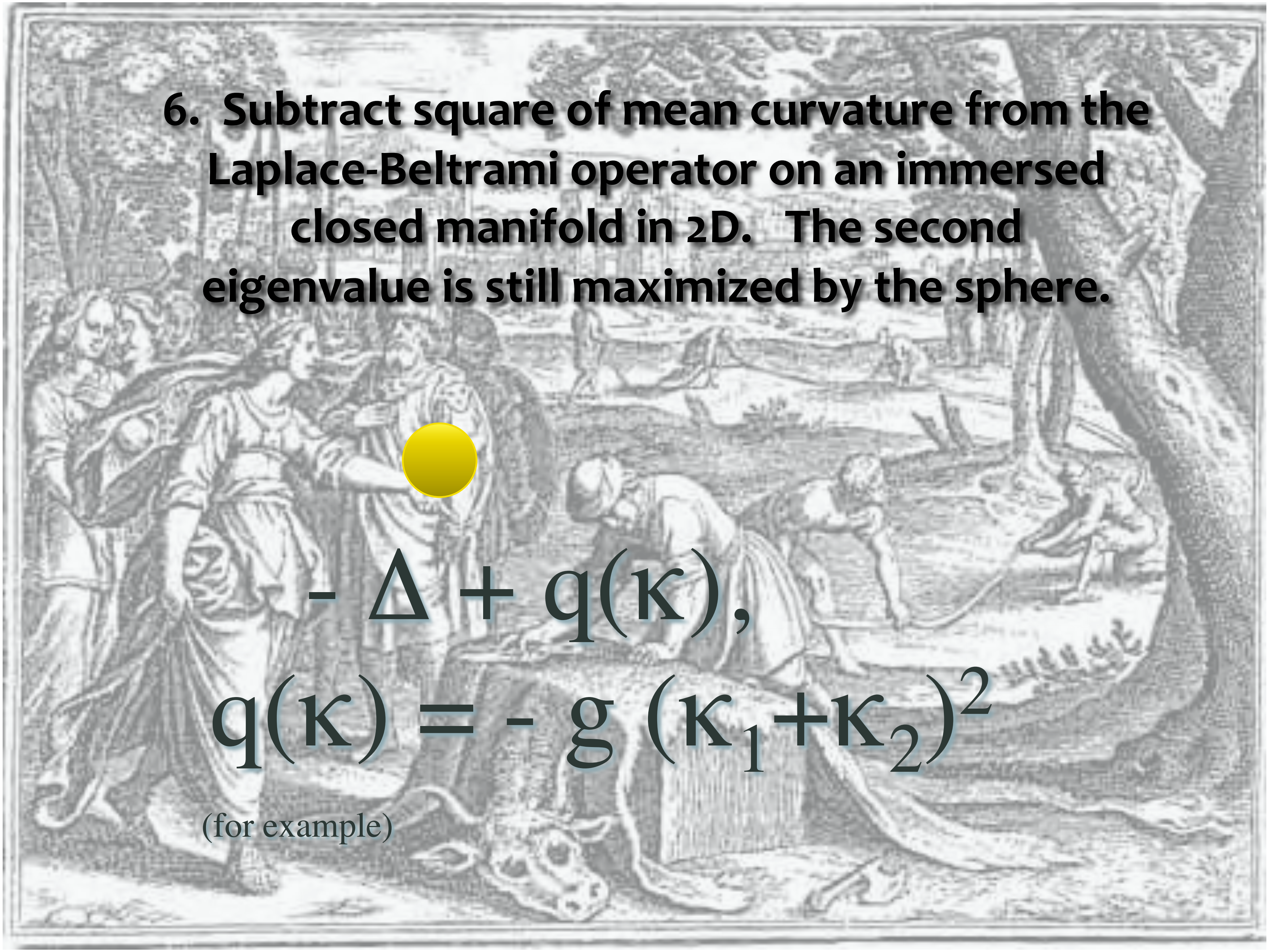
$$a \le \frac{b_j}{c_j}$$

$\Rightarrow$

$$a \le \frac{\sum_j b_j}{\sum_j c_j}:$$

$$\lambda_2 \le \frac{8\pi - g \int_\Omega \kappa_1 \kappa_2 dS}{\int_\Omega 1 dS} = \frac{8\pi - g \cdot 4\pi}{|\Omega|}$$

Equality iff sphere. Why?

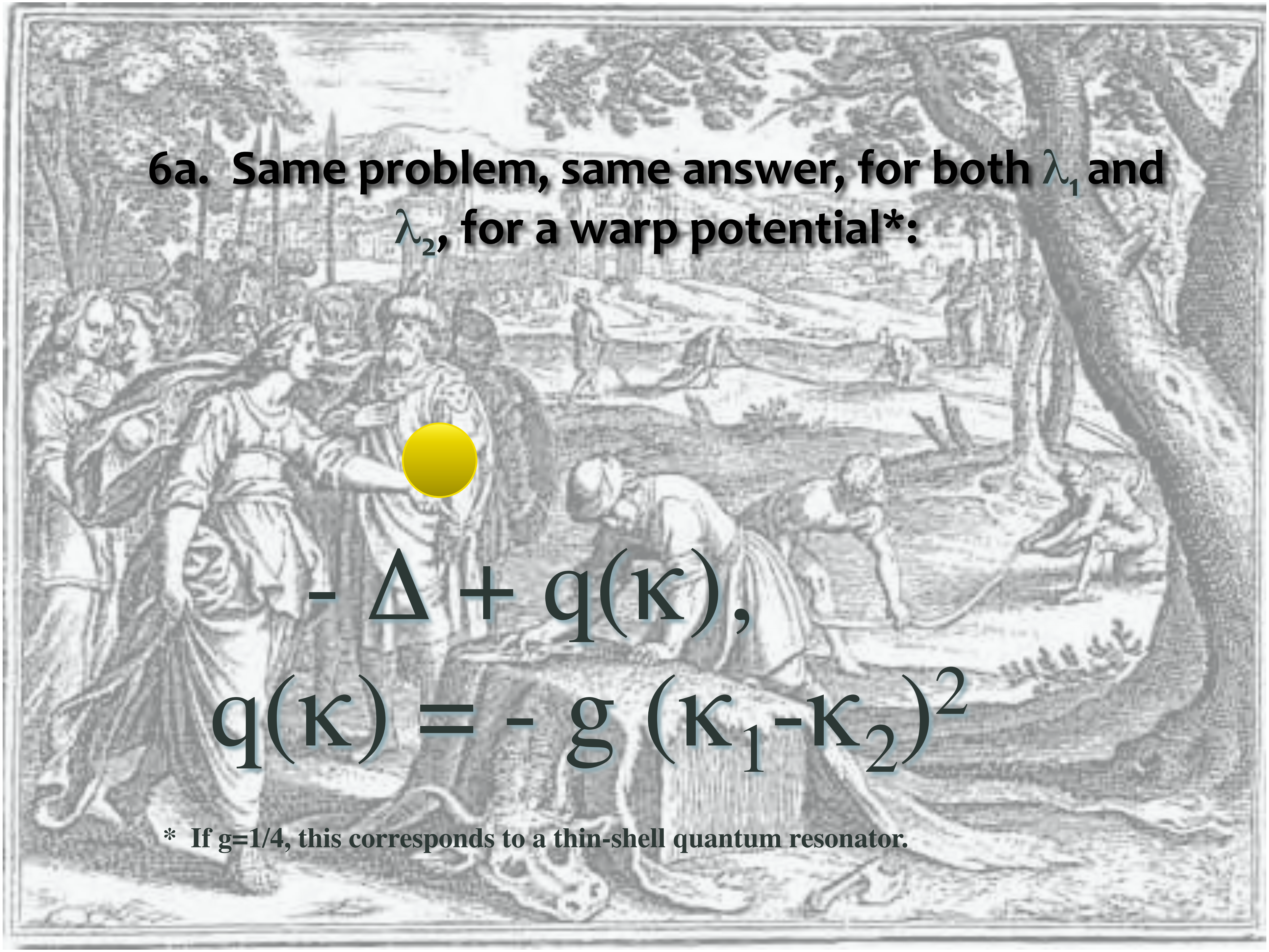


6. Subtract square of mean curvature from the Laplace-Beltrami operator on an immersed closed manifold in 2D. The second eigenvalue is still maximized by the sphere.

$$-\Delta + q(\kappa),$$

$$q(\kappa) = -\, g\, (\kappa_1 + \kappa_2)^2$$

(for example)



6a. Same problem, same answer, for both λ_1 and λ_2, for a warp potential*:

$$-\Delta + q(\kappa),$$

$$q(\kappa) = -g\,(\kappa_1 - \kappa_2)^2$$

* If g=1/4, this corresponds to a thin-shell quantum resonator.

7. Subtract square of mean curvature from the Laplace-Beltrami operator on an immersed closed manifold in *any* dimension. The second eigenvalue is still maximized by the sphere.

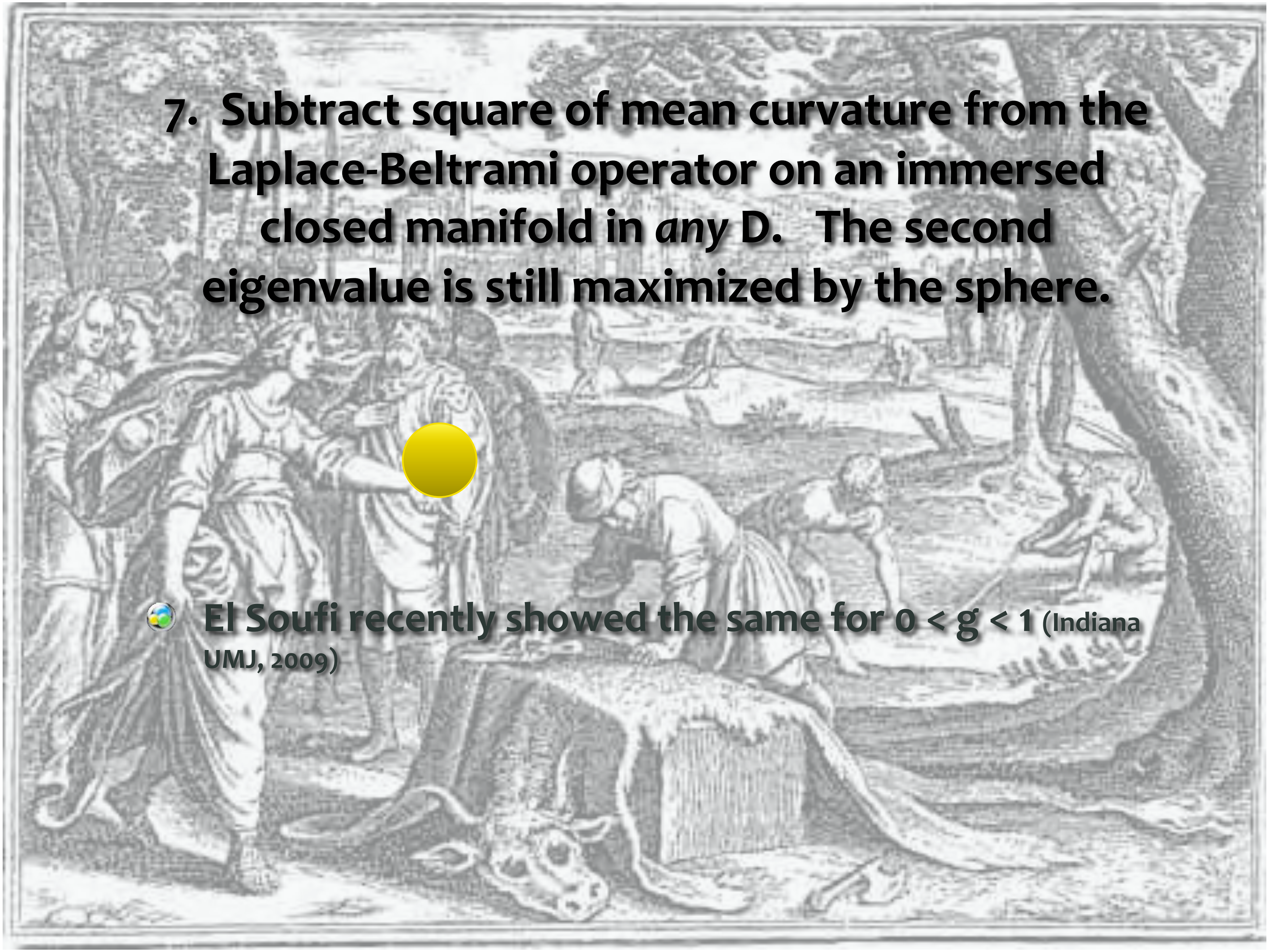
$$-\Delta + q(\kappa),$$

$$q(\kappa) = -g(\Sigma \kappa_j)^2$$

7. Subtract square of mean curvature from the Laplace-Beltrami operator on an immersed closed manifold in *any* D. The second eigenvalue is still maximized by the sphere.

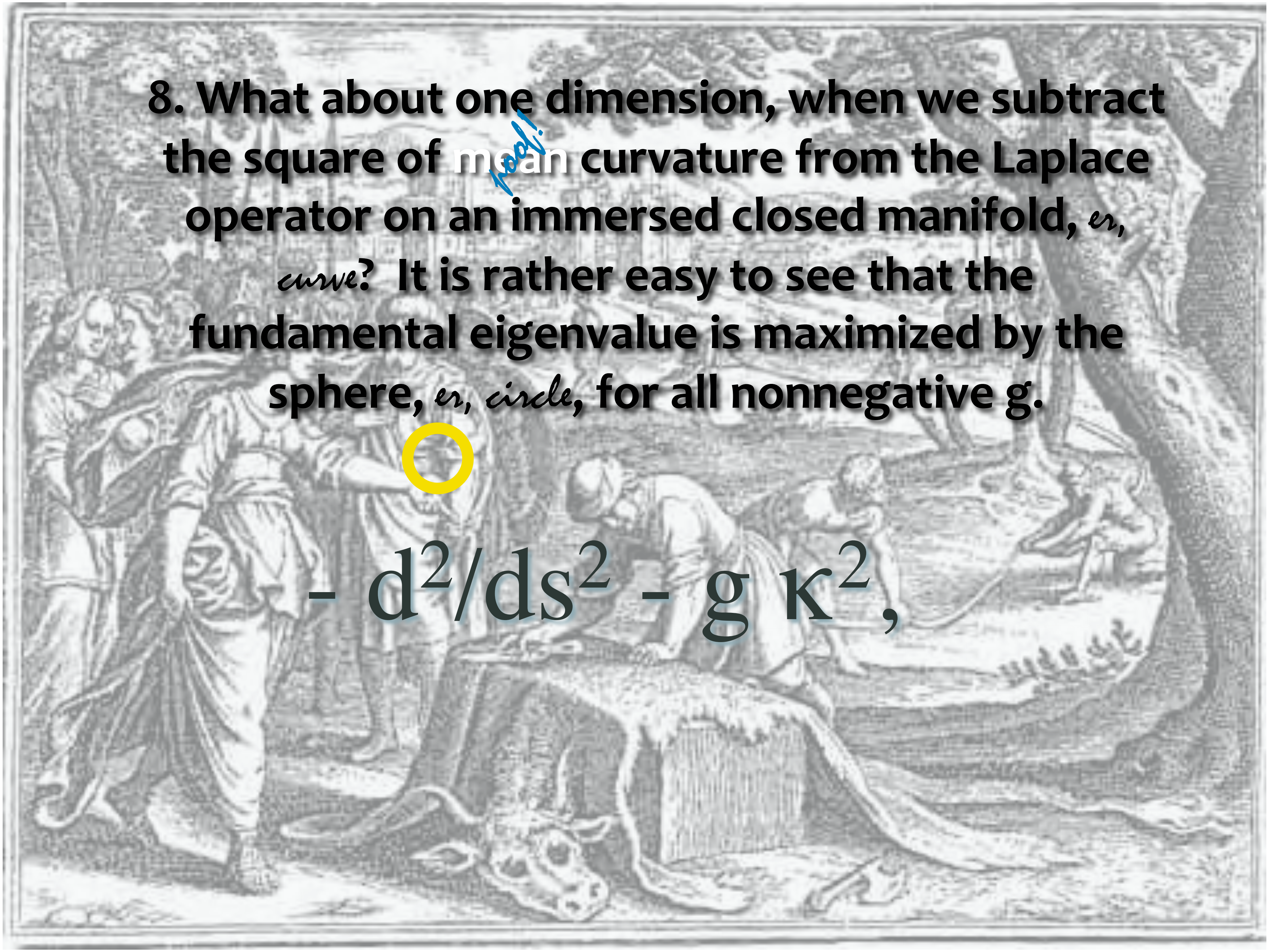
- The breakthrough was to use the Birman-Schwinger principle to count negative eigenvalues rather than estimating them, because, uniquely for the sphere,

$$\lambda_2 = 0$$



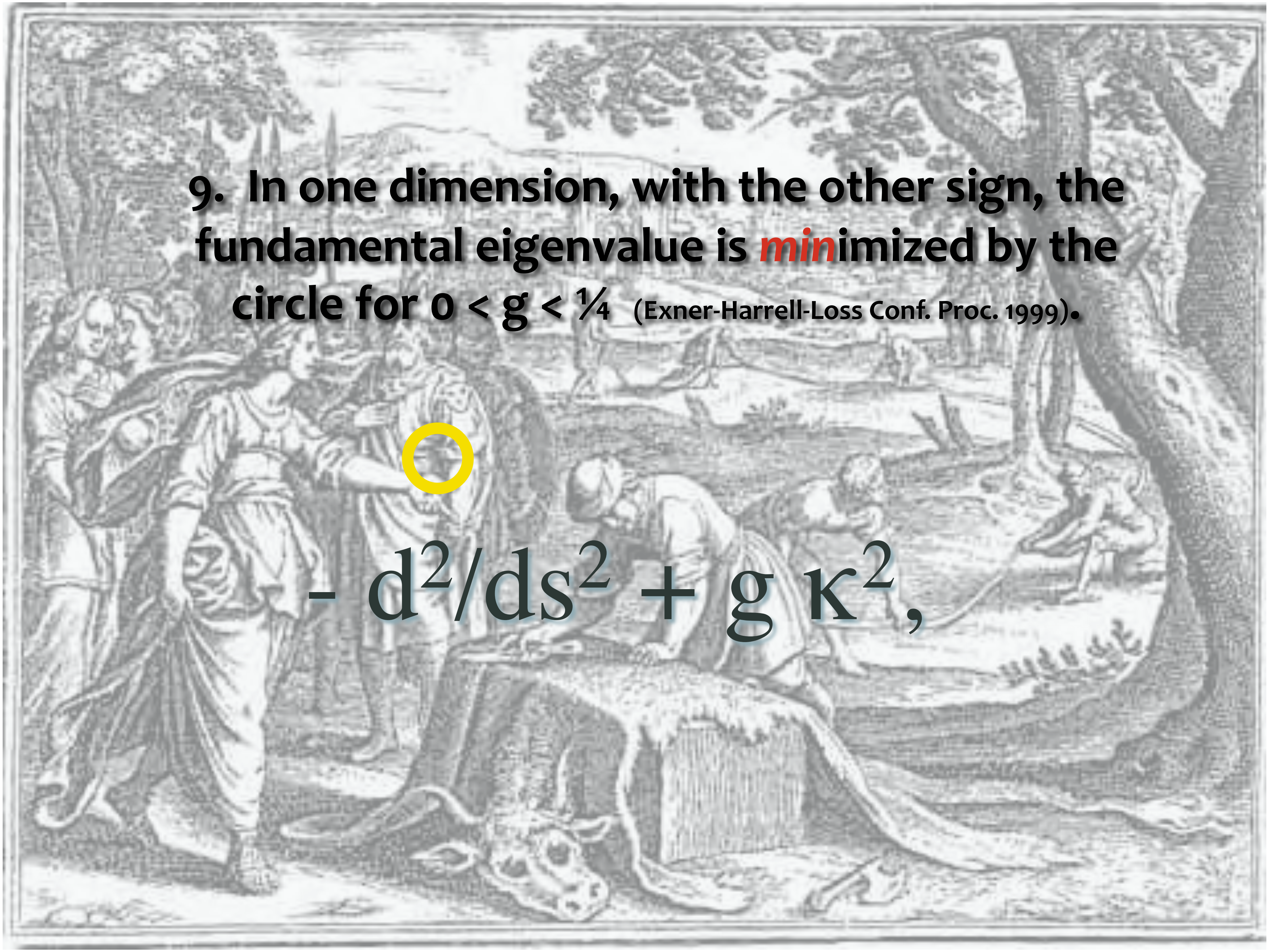
7. Subtract square of mean curvature from the Laplace-Beltrami operator on an immersed closed manifold in *any* D. The second eigenvalue is still maximized by the sphere.

- El Soufi recently showed the same for $0 < g < 1$ (Indiana UMJ, 2009)



8. What about one dimension, when we subtract the square of mean *poof!* curvature from the Laplace operator on an immersed closed manifold, *er, curve*? It is rather easy to see that the fundamental eigenvalue is maximized by the sphere, *er, circle*, for all nonnegative g.

$$- d^2/ds^2 - g\,\kappa^2,$$



9. In one dimension, with the other sign, the fundamental eigenvalue is *min*imized by the circle for 0 < g < ¼ (Exner-Harrell-Loss Conf. Proc. 1999).

$$-\, d^2/ds^2 + g\,\kappa^2,$$

9. In one dimension, with the other sign, the fundamental eigenvalue is *min*imized by the circle for $0 < g < 1/4$.

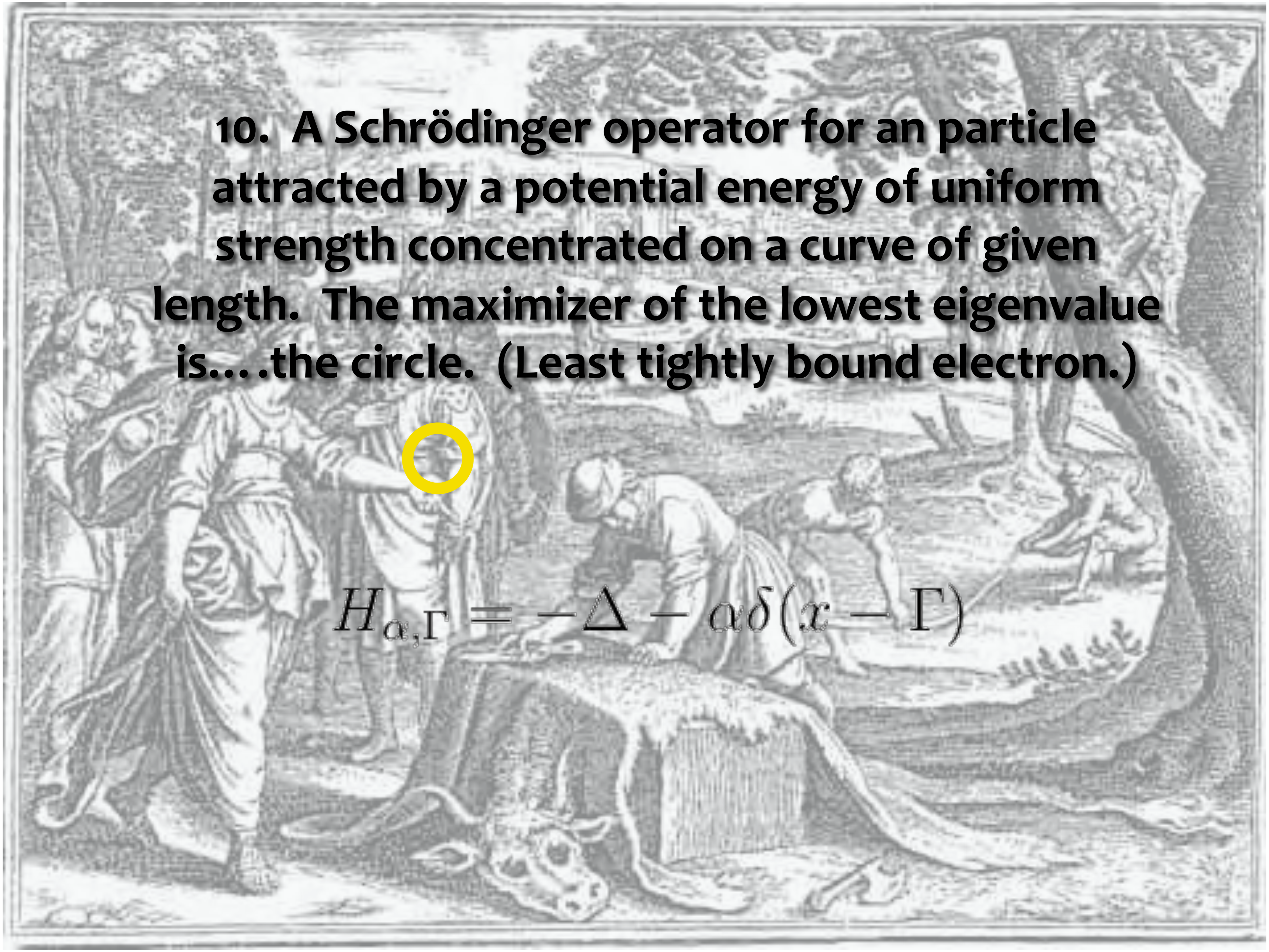
$$- d^2/ds^2 + g\,\kappa^2,$$

But this is not true for $g > 1$, and it is not very clear what happens between ¼ and 1!

9. In one dimension, with the other sign, the fundamental eigenvalue is *min*imized by the circle for 0 < g < 1/4.

$$- d^2/ds^2 + g\, \kappa^2 ,$$

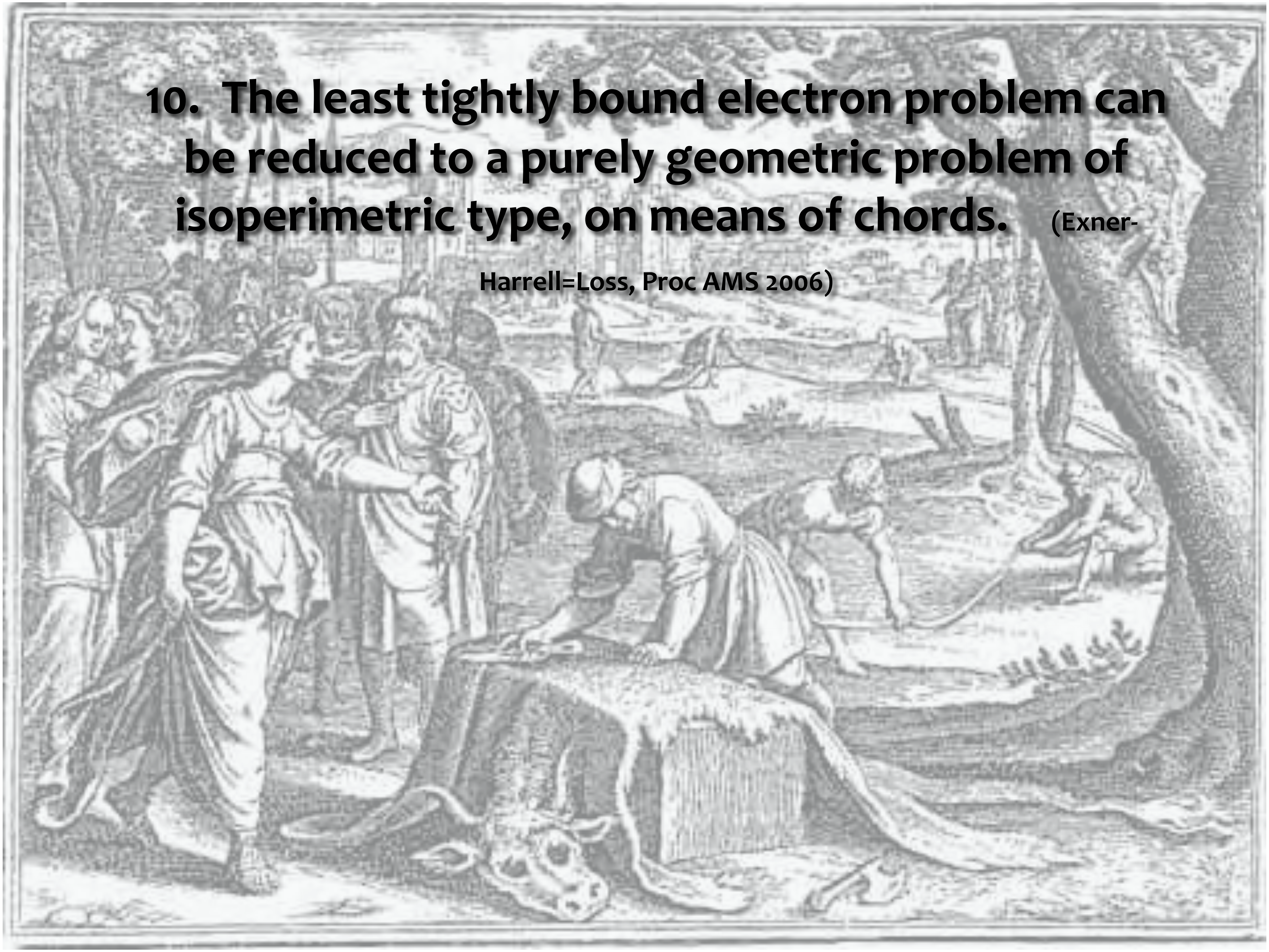
The conjecture is that there is a bifurcation at g=1, below which the circle is always the optimizer. (Remains open, some progress by Linde, Proc. AMS 2006.)



10. A Schrödinger operator for an particle attracted by a potential energy of uniform strength concentrated on a curve of given length. The maximizer of the lowest eigenvalue is….the circle. (Least tightly bound electron.)

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)$$

10. The least tightly bound electron problem can be reduced to a purely geometric problem of isoperimetric type, on means of chords. (Exner-Harrell=Loss, Proc AMS 2006)



11. The natural Schrödinger operator on a manifold from the point of view of *sum rules*:

$$-\Delta_{LB} + q,$$

where

$$q(\mathbf{x}) = \frac{1}{4}\left(\sum_{j=1}^{d} \kappa_j\right)^2$$

Commutators of operators

- $[G, [H, G]] = 2\ GHG - G^2H - HG^2$
- Etc., etc. Typical consequence:

$$\langle \phi_{\mathbf{j}}, [\mathbf{G}, [\mathbf{H}, \mathbf{G}]]\phi_{\mathbf{j}}\rangle = \sum_{\mathbf{k}:\lambda_{\mathbf{k}} \neq \lambda_{\mathbf{j}}} (\lambda_{\mathbf{k}} - \lambda_{\mathbf{j}})|\mathbf{G}_{\mathbf{kj}}|^{\mathbf{2}}$$

(Abstract version of Bethe’s sum rule)

1st and 2nd commutators

$$\frac{1}{2}\sum_{\lambda_j \in J} (z-\lambda_j)^2 \langle [G,[H,G]]\phi_j, \phi_j \rangle - \sum_{\lambda_j \in J} (z-\lambda_j) \| [H,G]\phi_j \|^2$$

$$=$$

$$\sum_{\lambda_j \in J} \sum_{\lambda_k \in J^c} (z-\lambda_j)(z-\lambda_k)(\lambda_k - \lambda_j) | \langle G\phi_j, \phi_k \rangle |^2$$

Harrell-Stubbe TAMS 1997

The only assumptions are that H and G are self-adjoint, and that the eigenfunctions are a complete orthonormal sequence.

1st and 2nd commutators

$$\frac{1}{2}\sum_{\lambda_j \in J} (z-\lambda_j)^2 \langle [G,[H,G]]\phi_j, \phi_j \rangle - \sum_{\lambda_j \in J} (z-\lambda_j) \| [H,G]\phi_j \|^2$$

$$=$$

$$\sum_{\lambda_j \in J} \sum_{\lambda_k \in J^c} (z-\lambda_j)(z-\lambda_k)(\lambda_k-\lambda_j) | \langle G\phi_j, \phi_k \rangle |^2$$

When does this side have a sign?

What you should remember about trace formulae/sum rules in a short seminar?

1. There is an exact identity involving traces including [G, [H, G]] and [H,G]*[H,G].

2. For the lower part of the spectrum it implies an inequality of the form:

$$\sum (z - \lambda_k)^2 (\ldots) \quad \leq \quad \sum (z - \lambda_k) (\ldots)$$

Canonical choice of G

A good choice of G for the Laplacian is a coordinate function, because

a) $[H,G] = -2\,\partial/\partial x_k$, and

b) $[G, [H, G]] = 2$

Quadratic sum rule with curvature

- A good choice of $G = x_k$, a Euclidean coordinate from R^d restricted to the submanifold.

- There are messy terms, but when you sum the trace identity over $k = 1...d$, magical cancellations occur.

- Since there are second derivatives of x_k, there is a curvature contribution that doesn't go away.

Quadratic sum rule with curvature

$$\sum (z - \lambda_k)_+^2 \leq \frac{4}{d} \sum (z - \lambda_k)_+ T_k,$$

where now

$$T_k := \left\langle \phi_k, \left(-\Delta + \frac{(\sum_j \kappa_j)^2}{4} \right) \phi_k \right\rangle$$

Quadratic sum rule with curvature

Special case: $z = \lambda_2$.

$$(\lambda_2 - \lambda_1)^2 \le \frac{4}{d}(\lambda_2 - \lambda_1)\lambda_1 + \frac{\|h\|_\infty^2}{4}$$

Quadratic sum rule with curvature

Sum rules imply universal bounds on eigenvalue gaps for Schrödinger operators on closed submanifolds in terms of the lower spectrum. Let

$$\delta := \sup_M \frac{(\sum \kappa_j)^2}{4} - V(\mathbf{x})$$

Quadratic sum rule with curvature

Sum rules imply universal bounds on eigenvalue gaps for Schrödinger operators on closed submanifolds in terms of the lower spectrum. Let

$$\delta := \sup_M \frac{(\sum \kappa_j)^2}{4} - V(\mathbf{x})$$

$$[\lambda_n, \lambda_{n+1}] \subseteq \left[\left(1 + \frac{2}{d}\right) \overline{\lambda_n} - \sqrt{D_n^\delta}, \left(1 + \frac{2}{d}\right) \overline{\lambda_n} + \sqrt{D_n^\delta} \right]$$

Quadratic sum rule with curvature

Sum rules imply universal bounds on eigenvalue gaps for Schrödinger operators on closed submanifolds in terms of the lower spectrum. Let

$$\delta := \sup_M \frac{(\sum \kappa_j)^2}{4} - V(\mathbf{x})$$

Simplest case is

$$\lambda_2 - \lambda_1 \leq \frac{4}{d}(\lambda_1 + \delta)$$

An interesting model

$$H_g := -\Delta + g\left(\sum_j \kappa_j\right)^2$$

Maximal gaps occur for the sphere. The sphere has degeneracies, but *every one* of its gaps is maximal from the point of view of sum rules.

An explicit calculation shows that the bound is sharp for the non-zero eigenvalue gaps of the sphere, for which all the eigenvalues are known and elementary [20]: For simplicity, assume that $d = 2, g = \frac{1}{4}$, and that M is the sphere of radius 1 embedded in $\mathbb{R}^3$. Then $h = 2, \sigma = 1$, and:

$$\lambda_1 = 1; \lambda_2 = \lambda_3 = \lambda_4 = 3; \ldots ; \lambda_{(m-1)^2+1} = \cdots = \lambda_{m^2} = m^2 - m + 1.$$

For $n = m^2$, the calculation shows that $\overline{\lambda_n} = \frac{n+1}{2}$, and $\overline{\lambda_n^2} = \frac{n^2+n+1}{3}$. Hence $D_n = n$, and b) informs us that

$$\begin{aligned} 2\overline{\lambda_{m^2}} - m = m^2 - m + 1 \leq \lambda_{m^2} = m^2 - m + 1 \\ \leq \lambda_{m^2+1} = m^2 + m + 1 \leq 2\overline{\lambda_{m^2}} + m = m^2 + m + 1, \end{aligned}$$

and thus λ_{m^2} equals the lower bound $2\overline{\lambda_{m^2}} - m$ and λ_{m^2+1} equals the upper bound $2\overline{\lambda_{m^2}} + m$.

Corollary 3.4. a) *Let H be as (2.1), with M a compact, smooth hypersurface. Then*

$$[\lambda_n, \lambda_{n+1}] \subseteq \left[\left(1+\frac{2}{d}\right)\overline{\lambda_n} - \sqrt{D_n^\delta}, \left(1+\frac{2}{d}\right)\overline{\lambda_n} + \sqrt{D_n^\delta}\right],$$

with

$$D_n^\delta := \frac{4}{d^2}\left(\left(\frac{dn+2}{2}\right)^2 \overline{\lambda_n}^2 + (dn-d+2)\delta\overline{\lambda_n} - d\left(\frac{dn+4}{4}\right)\overline{\lambda_n^2} + \delta^2\right).$$

b) *For H_σ, $0 < \sigma$, of the form (1.10) on a smooth, compact hypersurface M,*

$$[\lambda_n, \lambda_{n+1}] \subseteq \left[\left(1+\frac{2\sigma}{d}\right)\overline{\lambda_n} - \sqrt{D_n}, \left(1+\frac{2\sigma}{d}\right)\overline{\lambda_n} + \sqrt{D_n}\right],$$

with

$$D_n := \left(\left(1+\frac{2\sigma}{d}\right)\overline{\lambda_n}\right)^2 - \left(1+\frac{4\sigma}{d}\right)\overline{\lambda_n^2} \geq 0.$$

This bound is sharp for every non-zero eigenvalue gap of H_1 on the sphere.

`You may seek it with trial functions---and seek it with care;
You may hunt it with rearrangements and hope;
You may perturb the boundary with a lump here and there;
You may fool it with some algebraic rope-a-dope---'

Apologies to Lewis Carroll.

The End

The image of Dido founding Carthage by Mathäus Merian the Elder (1593-1650) is in the public domain, according to:

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