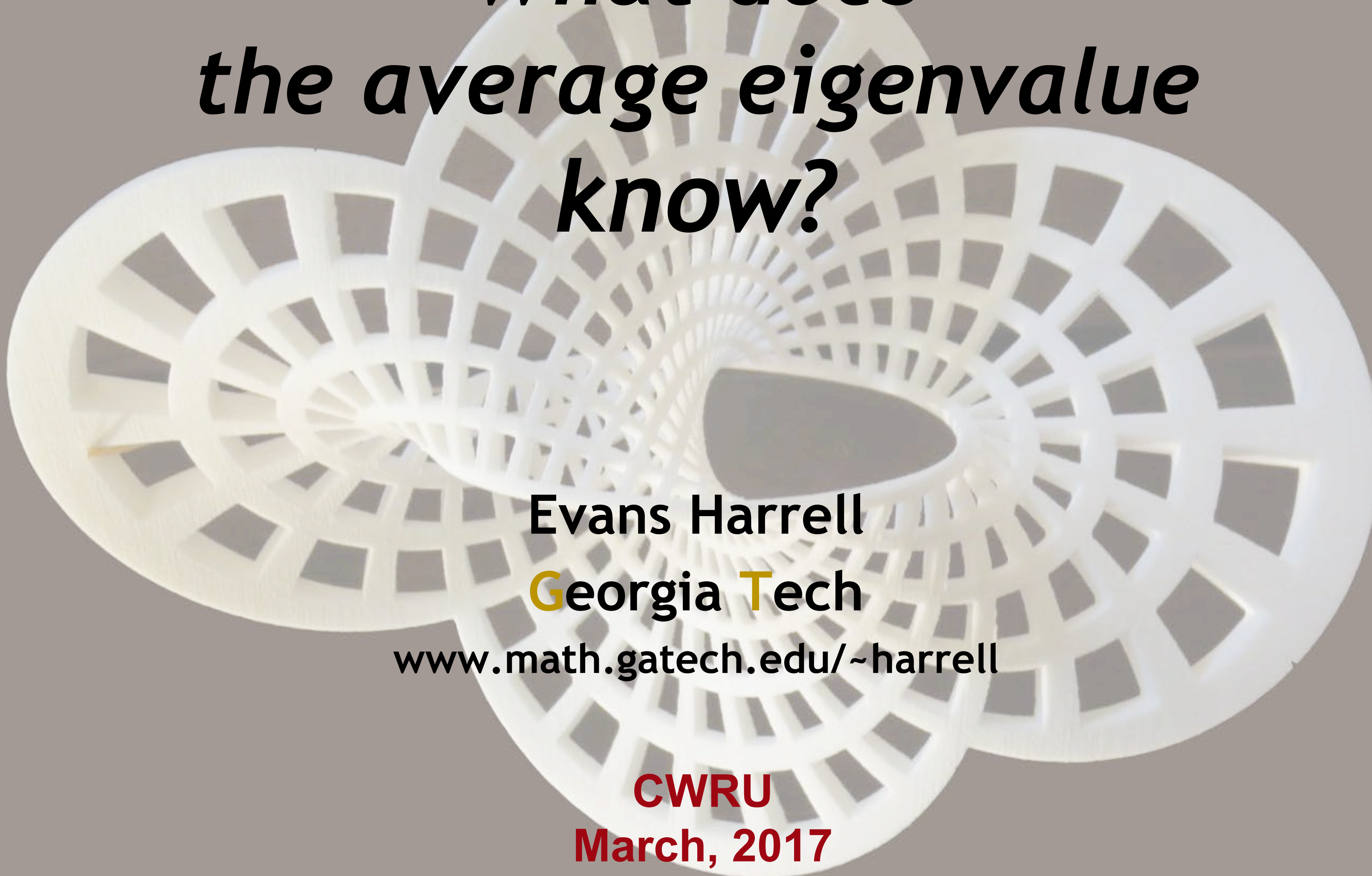

What does the average eigenvalue know?

Evans Harrell

Georgia Tech

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CWRU
March, 2017

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What does the average eigenvalue know about geometry, physics, or the complexity of a graph?

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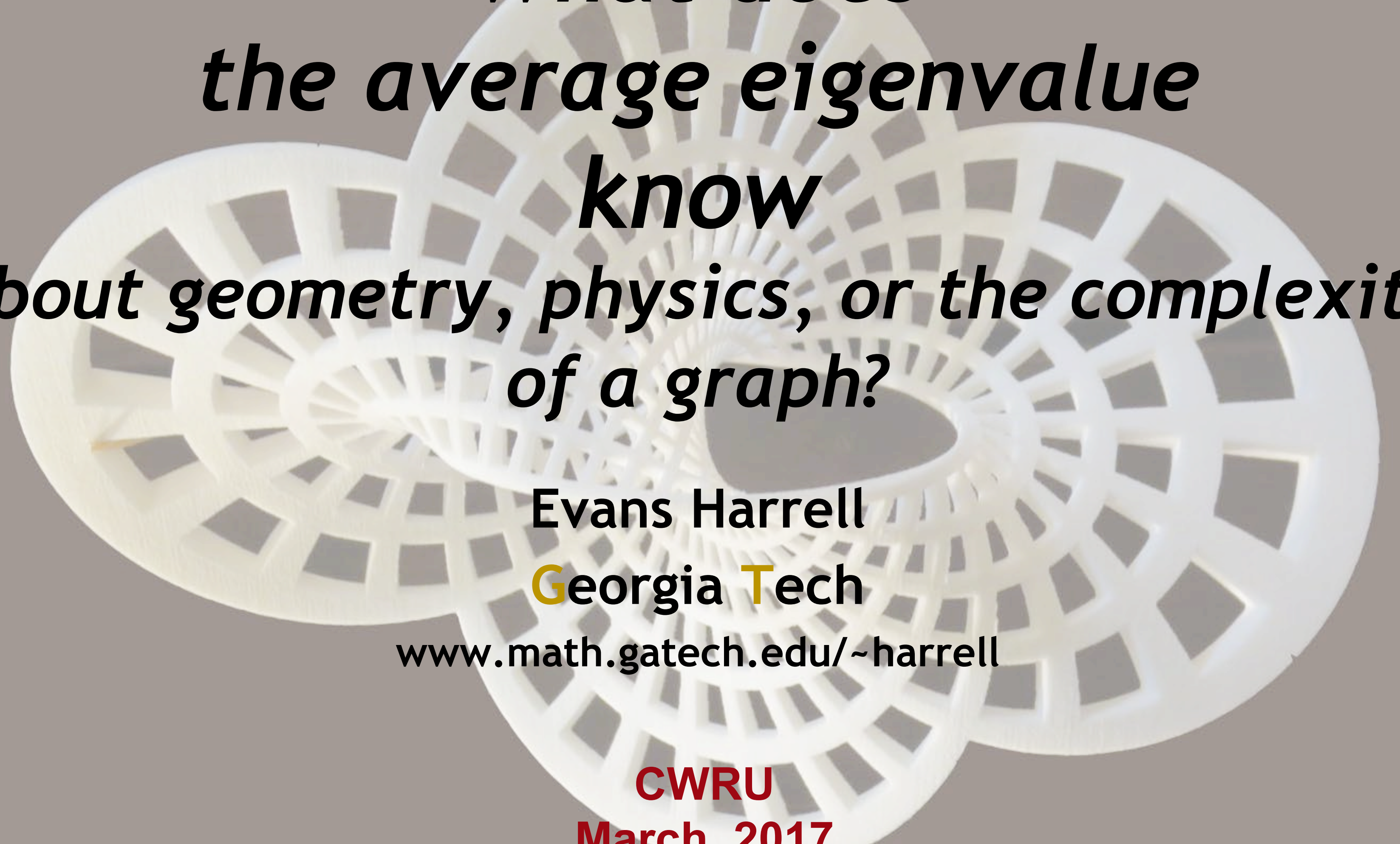

Thanks for the invitation!

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Abstract

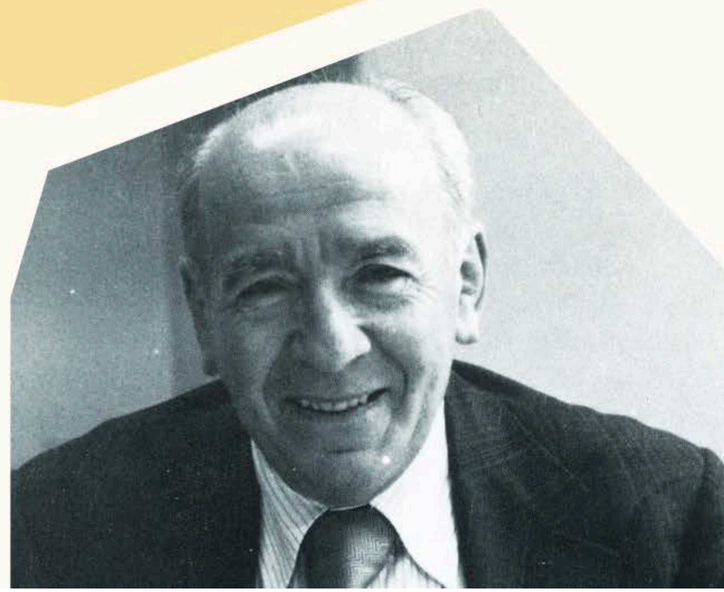
- Abstract. Inverse spectral theory refers to the use of eigenvalues and related information to understand a linear operator. Linear operators and matrices are used, for example, to describe such things as vibrating membranes, quantum phenomena, and graphs (in the sense of networks). The eigenvalues respond to the shape of the membrane, the form of the interaction potential, or respectively the connectedness of the graph, but not in a formulaic way. The effort to tease out these details from the knowledge of the eigenvalue spectrum was memorably described al little over 50 years ago by Mark Kac in an article entitled "Can one hear the shape of a drum?" (Incidentally, the answer is "Often, but not always.")

- In this lecture I will give some perspectives on inverse spectral problems and then concentrate on what can be learned from the statistical distribution of the eigenvalues, such as means, deviations, and partition functions. Much of this work is joint with J. Stubbe of EPFL.

Fifty years of hearing drums

Spectral Geometry and the legacy of Mark Kac

Santiago de Chile
May 16th-20th, 2016



M. Kac, Can one hear the shape of a drum?, Amer. Math. Monthly, 1966.

- $-\Delta u = (\omega/c)^2 u =: \lambda u$, with “Dirichlet-type” boundary conditions.

Borg in 1946 and then the school of Gel'fand had earlier considered the question of whether one could hear the density of a guitar string,

- $-u'' + q(x)\, u = \lambda\, u,$

but they failed to find as charming a catch phrase –

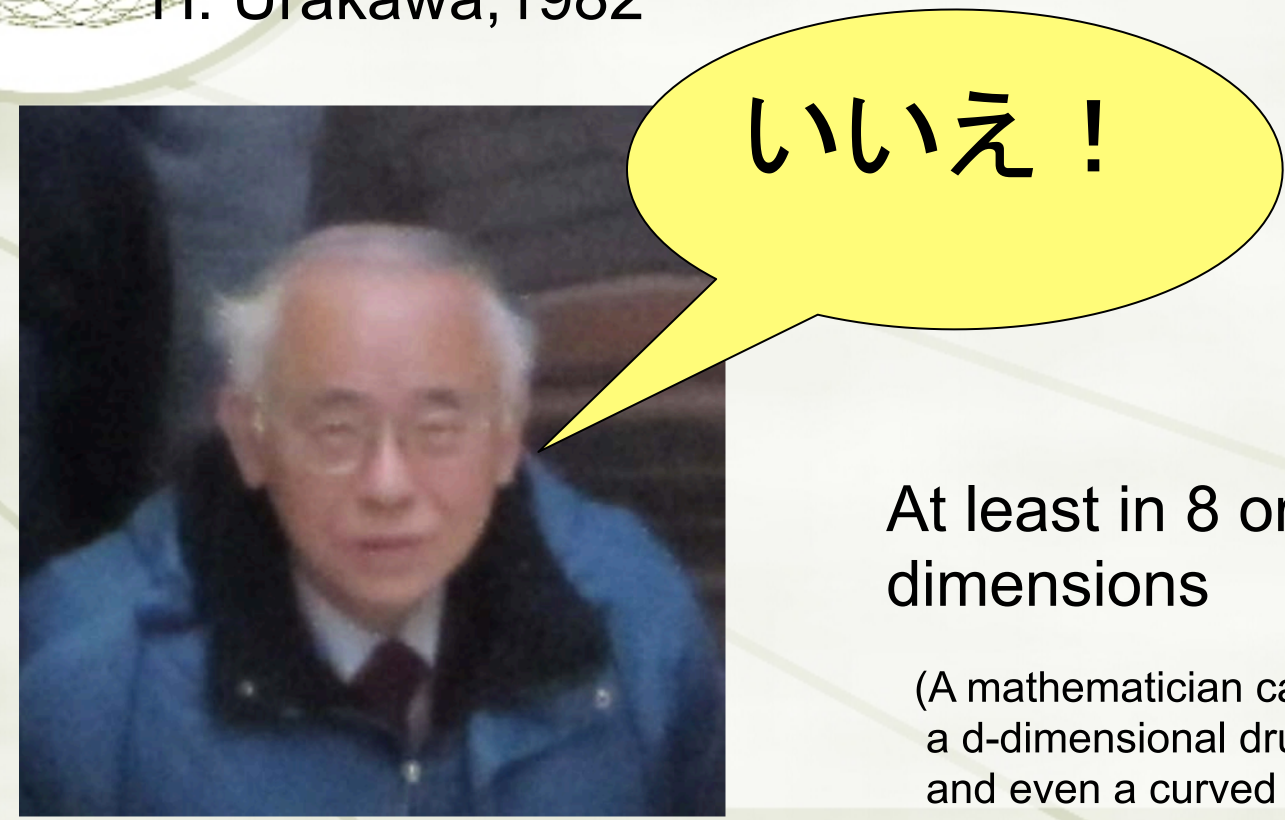
(Which was due to Bers!)

Well, can one hear the shape of a drum?

Circular ones, sure. No other drum has the same "spectrum" as the circular drum of a given area. So, the answer is, at least sometimes.

Well, can one hear the shape of a drum?

H. Urakawa,1982



At least in 8 or more dimensions

(A mathematician can of course play a d-dimensional drum, d arbitrary, and even a curved manifold.)

Well, can one hear the shape of a drum?

Gordon, Webb, and Wolpert, 1991. They used ideas of Sunada 1985.

We can hear the shape of a *sufficiently smooth* drum

- Zelditch, *Ann. Math.*, 2009. Given an analytic boundary, no holes, a line of reflection, and a condition on the closed billiard trajectories, …, yes, the drum is uniquely determined by the spectrum!

Some features are "audible"

- You can hear the area of the drum, by the *Weyl asymptotics*:
- For the drum problem

  $$\lambda_k \sim C_d \, (k/\mathrm{Vol}(\Omega))^{2/d}.$$

- Notice that in addition to the volume,we can hear the dimension.

Well, can one hear the shape of a drum?

Extreme cases are often unique. For example, what drum (of a given area or in a given category, such as convex) maximizes or minimizes λ_1 or some other λ_k?

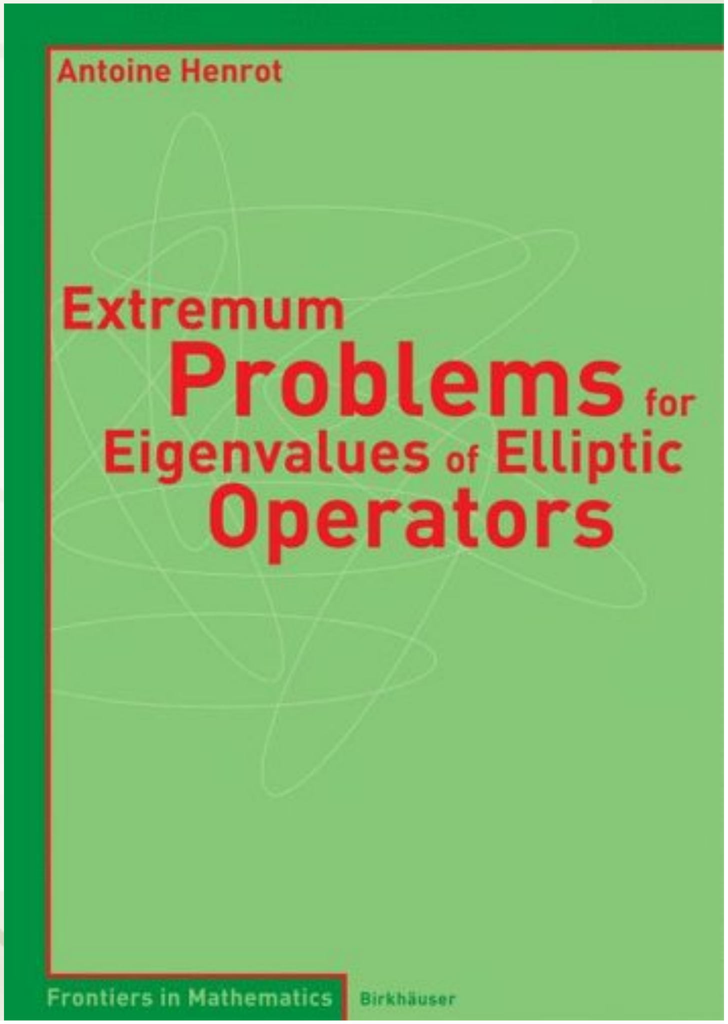
Some features are "audible"

- The Faber-Krahn theorem states that the dominant eigenvalue is minimized by the circular drum (area fixed).

  $$\lambda_1 \geq (\lambda_1(B_{1,d}))\,(\mathrm{Vol}(B_{1,d})/\mathrm{Vol}(\Omega))^{2/d}.$$

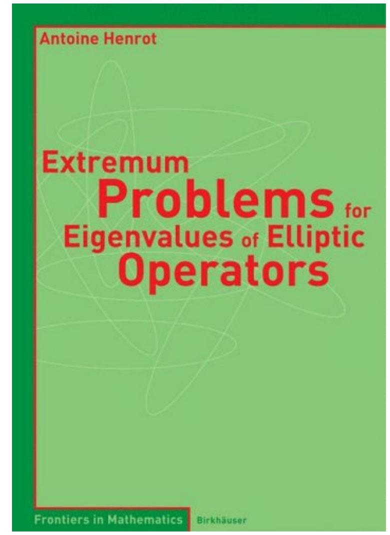
- So from the dominant eigenvalue and the top of the spectrum, you can tell whether the drum is circular.

To extremists, things tend to sound simple…



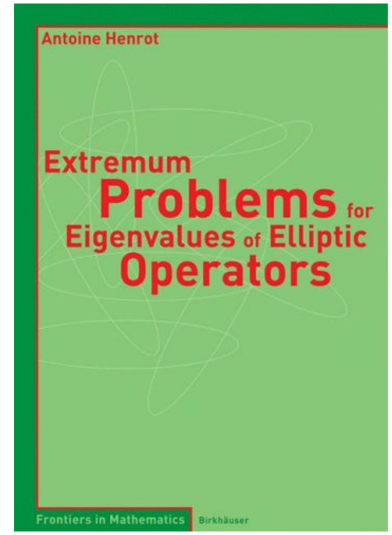
Is the extremum always a union of round shapes?

5.4. *Case of higher eigenvalues* 83



No	Optimal union of discs		Computed shapes	
3		46.125		46.125
4		64.293		64.293

Is the extremum always a union of round shapes?



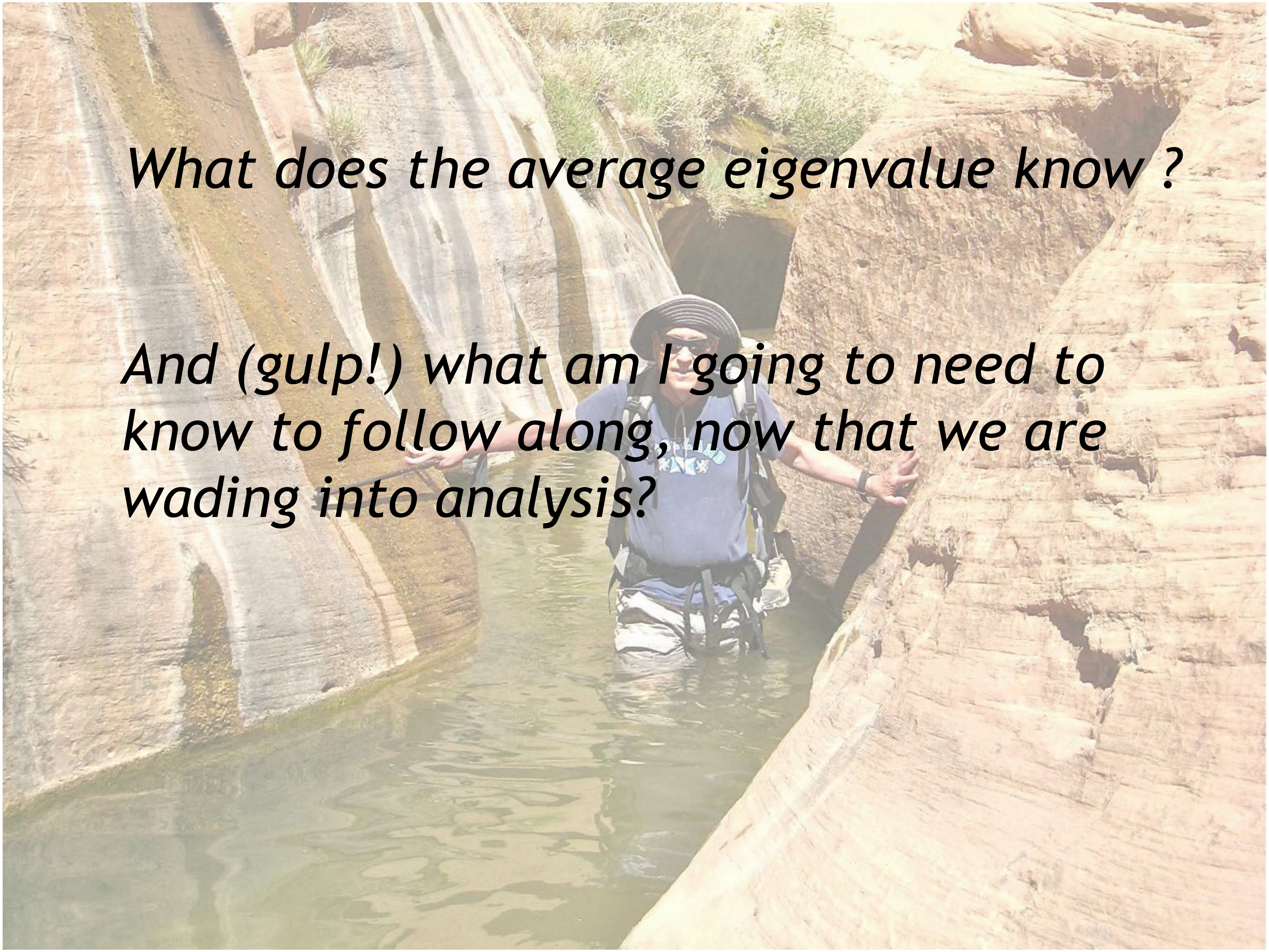
No	Optimal union of discs	Computed shapes
3	46.125	46.125
4	64.293	64.293
5	82.462	**78.47**
6	92.250	**88.96**
7	110.42	**107.47**
8	127.88	**119.9**
9	138.37	**133.52**
10	154.62	**143.45**

Other inverse spectral problems

- Manifolds
- Graphs
- Quantum (metric) graphs

Averages of eigenvalues

- What's so great about averages?
- And what does the average eigenvalue know?


What does the average eigenvalue know ?

And (gulp!) what am I going to need to know to follow along, now that we are wading into analysis?

Mathematical background

1. You can find eigenvalues "variationally."
2. A few words about graphs.
3. As some stage I'll use the Fourier transform, and the main thing you need is the Parseval relation. I will also mention "tight frames," which enjoy a similar property.

Variational eigenvalues

The eigenvalues of a self-adjoint operator (like a symmetric matrix) are are determined by a min-max procedure. The lowest one satisfies the Rayleigh-Ritz inequality,

$$\lambda_0 = \min_{\|\phi\|=1} \langle \phi, A\phi \rangle$$

while the other ones satisfy a min-max formula involving orthogonalization.

Variational eigenvalues

Under the same circumstances as in the min-max principle, suppose that $\{\phi_1, \ldots, \phi_\ell\}$ is an orthonormal set of functions in the quadratic-form domain of H. Prove the *variational principle for sums*,

$$\sum_{j=1}^{\ell} \lambda_j^{\uparrow} \leq \sum_{j=1}^{\ell} E_H(\phi_j),$$

Averages of eigenvalues

- Suppose that you know about

$$\overline{\mu_k} := \frac{1}{k} \sum_{j=0}^{k-1} \mu_j$$

(say, upper or lower bounds). Then you know about several other amusing quantities.

Averages of eigenvalues

- It is a small step from sums to *Riesz means*,

$$R_\sigma(z) := \sum_\ell (z - \mu_\ell)_+^\sigma$$

In particular,

$$R_1(z) = z\mathcal{N}[z] - S_{[z]}$$

$$S_k := \sum_{\ell=0}^{k-1} \mu_\ell$$

Averages of eigenvalues

- Classical transforms can more directly convert bounds on Riesz means

$$R_\sigma(z) := \sum_\ell (z - \mu_\ell)_+^\sigma$$

(pretty much equivalent to sums) into bounds on a class of functions including the spectral partition and zeta functions.

$$Z(t) := \sum e^{-\mu_k t} \qquad \zeta(s) := \sum \mu_k^{-s}$$

Averages of eigenvalues

- With the Laplace transform,

$$\mathcal{L}\left((z-\mu_\ell)_+^\sigma\right) = \frac{\Gamma(\sigma+1)e^{-\mu_\ell t}}{t^{\sigma+1}}$$

Averages of eigenvalues

- With the Laplace transform,

$$\mathcal{L}\left((z-\mu_\ell)_+^\sigma\right) = \frac{\Gamma(\sigma+1)e^{-\mu_\ell t}}{t^{\sigma+1}}$$

so knowing about sums means knowing about the spectral partition function (= "trace of the heat kernel), which connects to the spectral ζ function.

Karamata's theorem

A theorem of Karamata provides similar information without passing through Riesz means and transforms.

(Karamata is a strengthening of Jensen if you have additional information about ordering and domination.)

Averages of eigenvalues

- With Karamata's theorem, a bound of the form $\sum_{j=0}^{k-1} \mu_j \leq \sum_{j=0}^{k-1} m_j$ implies that $\sum_{j=0}^{k-1} \Psi(\mu_j) \geq \sum_{j=0}^{k-1} \Psi(m_j)$ for decreasing convex functions (e.g., the spectral partition and zeta functions).

Karamata's theorem

Lemma 3.1 (Karamata-Ostrowski) *Let two nondecreasing ordered sequences of real numbers* $\{\mu_j\}$ *and* $\{m_j\}$, $j = 0, \ldots, n-1$, *satisfy*

$$\sum_{j=0}^{k-1} \mu_j \leq \sum_{j=0}^{k-1} m_j \tag{3.7}$$

for each k. *Then for any differentiable convex function* $\Psi(x)$,

$$\sum_{j=0}^{k-1} \Psi(\mu_j) \geq \sum_{j=0}^{k-1} \Psi(m_j) + \Psi'(m_{k-1}) \cdot \sum_{j=0}^{k-1} (\mu_j - m_j).$$

In particular, assuming either that Ψ *is nonincreasing or that* $\sum_{j=0}^{k-1} \mu_j = \sum_{j=0}^{k-1} m_j$,

$$\sum_{j=0}^{k-1} \Psi(\mu_j) \geq \sum_{j=0}^{k-1} \Psi(m_j)$$

Combinatorial graphs

- At its most abstract, a network is a graph, consisting of “vertices” (or “nodes”) and information about which are connected to which.

Combinatorial graphs

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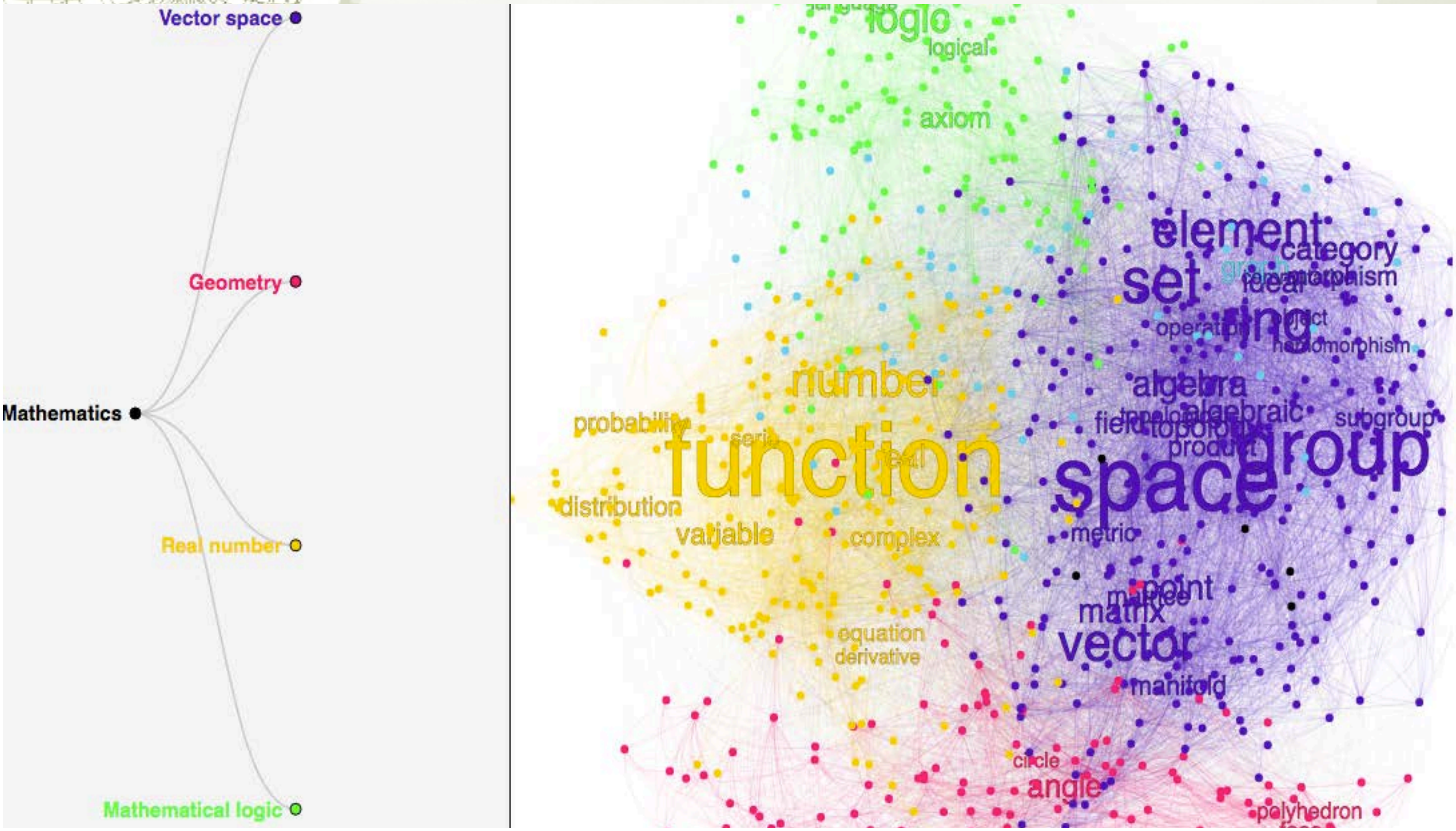
- The “adjacency matrix” has entries $a_{uv}=1$ when u is connected to v, and otherwise 0.

Combinatorial graphs

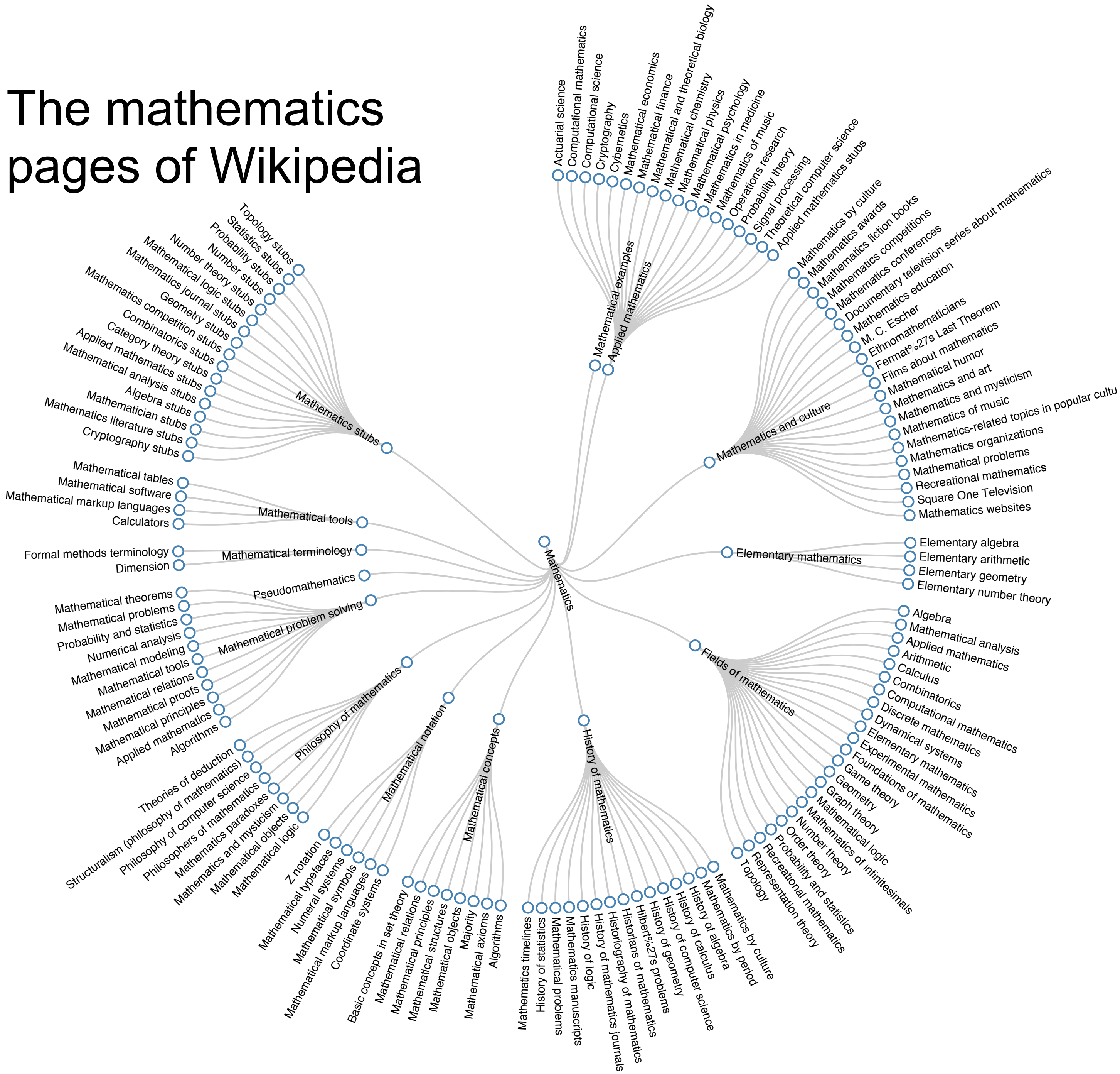
- At its most abstract, a network is a graph, consisting of “vertices” (or “nodes”) and information about which are connected to which.
- The “adjacency matrix” has entries $a_{uv}=1$ when u is connected to v, and otherwise 0.
- Everybody’s favorite networks are the internet and subsets like Facebook and Wikipedia, so …..

Undergraduate research project of Philippe Laban
See http://wikigraph.gatech.edu/

Mathematics in the wikipedia



The mathematics pages of Wikipedia



The "adjacency matrix" for PDEs in wikipedia

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   0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1,
   0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1},
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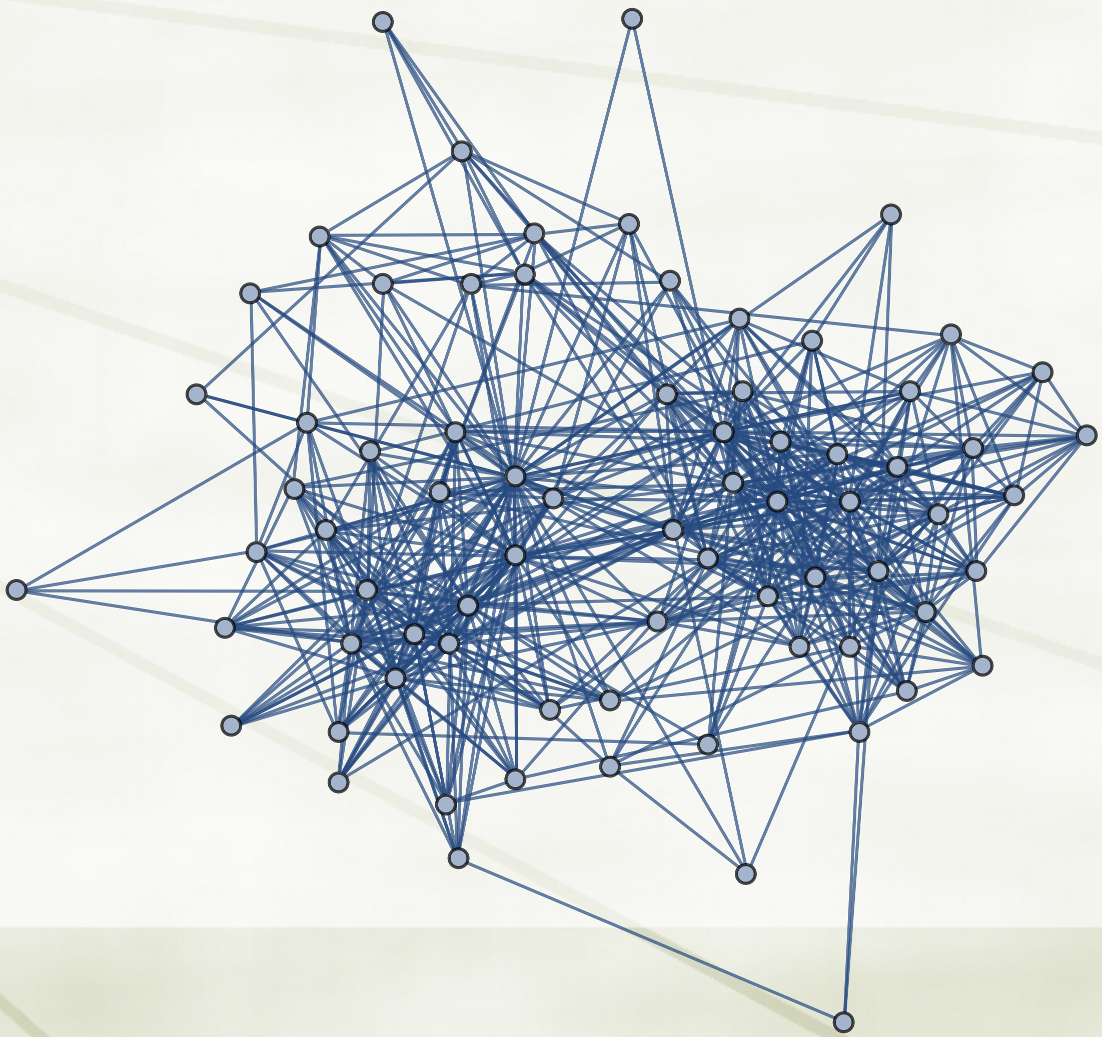
The adjacency matrix for PDEs in wikipedia



The adjacency matrix for Analysis

The adjacency matrix for Graph Theory

Combinatorial graphs

- A graph connects n vertices with edges as specified by an adjacency matrix A, with a_{ij} = 1 when i and j are connected, otherwise 0. The graph is not a priori living in Euclidean space.
- Discretizations of domains are graphs, but graphs are far more general.
- Spectral techniques have been known since the 1970's to reflect the structure of a graph (Fiedler, Chung), but not to determine it in general.

Combinatorial graphs

- A graph can be described not only via the adjacency matrix A but by a selection of other reasonable symmetric real matrices, notably the graph Laplacian.
- You can access my lectures from a recent CIMPA school in southern Tunisia at

  mathphysics.com

Combinatorial graphs

- The *graph Laplacian* is a matrix that compares values of a function at a vertex with the average of its values at the neighbors.

  H := -Δ := Deg – A, where

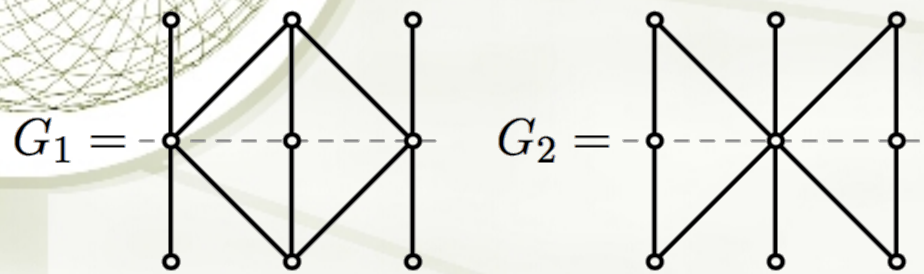
  Deg := diag(d_v), d_v := # neighbors of v.

  Its weak form is:

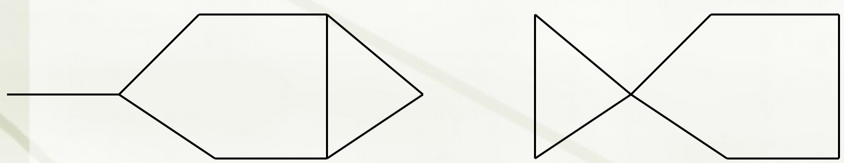
$$f \to \frac{1}{2} \sum_u \sum_{v \sim u} |f_u - f_v|^2$$

Co-spectral graphs.

Example of Steve Butler, Iowa State, for the adjacency matrix (and the “normalized Laplacian”)



Mouse and fish for the standard Laplacian.



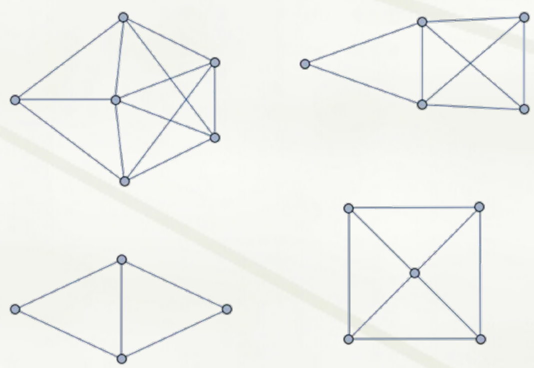
There are different ways to set up the discrete Laplacian, but in any version, the frequencies do not always determine the graph.

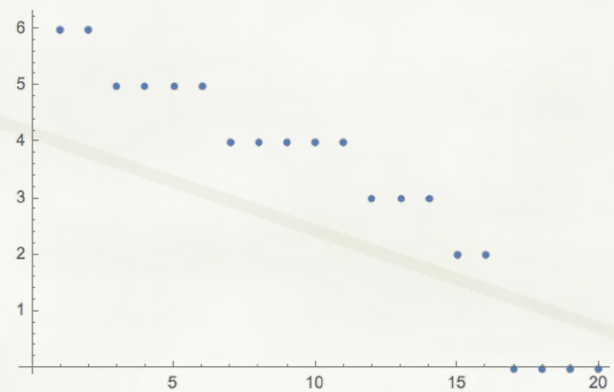
Information contained in graph spectra

- By sp(H):
  - Number of connected components (= dim of 0 eigenspace)
  - Number of edges = Tr(H)/2
- Not by sp(H) but by sp(A):
  - Number of triangles
  - Colorings. Bipartite graphs are easily identifiable by their spectra.

Information contained in graph spectra

- Fiedler, 73, "Community detection," second eigenfunction of H partitions graph into two clusters, or communities. This generalizes to multiple clusters.





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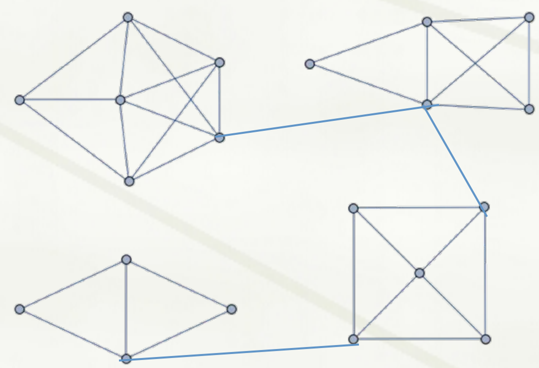
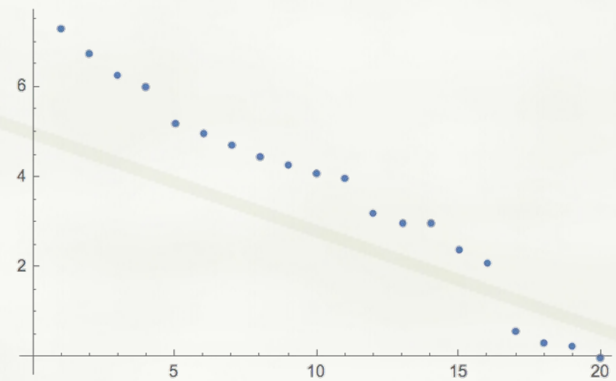




Graph analogues of Weyl-type bounds, and dimensionality

For Laplacians (DBC or NBC):

- Weyl law $$\lambda_k \sim 4\pi^2 \left(\frac{k}{C_d|\Omega|}\right)^{2/d}$$
- Berezin-Li-Yau $$\overline{\lambda_k} \leq \frac{d}{d+2} \frac{4\pi^2}{C_d} \left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}}$$
- Kröger $$\overline{\mu_k} \geq \frac{d}{d+2} \frac{4\pi^2}{C_d} \left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}}$$

A new hammer in search of nails

(It’s just an average hammer.)

The average hammer

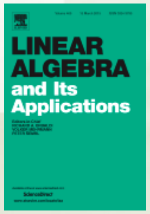
$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \le \frac{1}{|\mathfrak{M}_0|}\int_{\mathfrak{M}_0}\frac{Q_M(f_\zeta, f_\zeta)}{\|f_\zeta\|^2}\,d\sigma$$

where

$$\int_{\mathfrak{M}}\frac{|\langle\phi, f_\zeta\rangle|^2}{\|f_\zeta\|^2}\,d\sigma = A\|\phi\|^2$$

for a fixed constant $A > 0$, *and* $\mathfrak{M}_0 \subset \mathfrak{M}$ *such that* $|\mathfrak{M}_0| \geq kA$.

Harrell-Stubbe LAA, 2014

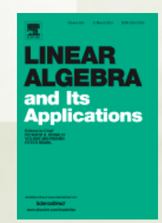


The average hammer

$$\frac{1}{k}\sum_{j=0}^{k-1}\mu_j \le \frac{1}{|\mathfrak{M}_0|}\int_{\mathfrak{M}_0}\frac{Q_M(f_\zeta, f_\zeta)}{\|f_\zeta\|^2}\,d\sigma$$

Averages within averages!

Harrell-Stubbe LAA, 2014



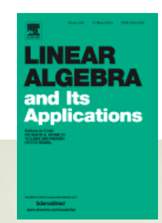
The averaged variational principle for sums

Theorem 3.1 *Consider a self-adjoint operator M on a Hilbert space $\mathcal{H}$, with ordered, entirely discrete spectrum $-\infty < \mu_0 \leq \mu_1 \leq \dots$ and corresponding normalized eigenvectors $\{\psi^{(\ell)}\}$. Let f_ζ be a family of vectors in $\mathcal{Q}(M)$ indexed by a variable ζ ranging over a measure space $(\mathfrak{M}, \Sigma, \sigma)$. Suppose that $\mathfrak{M}_0$ is a subset of $\mathfrak{M}$. Then for any eigenvalue μ_k of M,*

$$\begin{aligned}&\mu_k\left(\int_{\mathfrak{M}_0} \langle f_\zeta, f_\zeta\rangle \, d\sigma - \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)}\rangle|^2 \, d\sigma\right) \\ &\leq \\ &\int_{\mathfrak{M}_0} \langle H f_\zeta, f_\zeta\rangle d\sigma - \sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)}\rangle|^2 \, d\sigma,\end{aligned} \tag{3.2}$$

provided that the integrals converge.

Harrell-Stubbe LAA, 2014



Implications for Riesz means

- If there is a generalized Parseval identity (as with a *tight frame*):

$$\int_{\mathfrak{M}} |\langle f_\zeta, \phi \rangle|^2 d\sigma = C\|\phi\|^2,$$

then

$$\sum_j (z - \lambda_j)_+ \geq \frac{1}{C} \int_{\mathfrak{M}_0} \left(z\|f_\zeta\|^2 - Q_M(f_\zeta, f_\zeta) \right) d\sigma.$$

Recent applications of the averaged variational principle:

1. Harrell-Stubbe, *LAA* 2014: Weyl-type upper bounds on sums of eigenvalues of (discrete) graph Laplacians and related operators.
2. El Soufi-Harrell-Ilias-Stubbe, *J. Spectral Theory*, to appear: Semiclassically sharp Neumann bounds for a large family of 2^{nd} order PDEs.
3. Harrell-Stubbe, Two-term, Weyl estimates for eigenvalue means of the Laplacian, *J. Spectral Theory*, to appear.

Combinatorial graphs

- A graph connects n vertices with edges as specified by an adjacency matrix A, with $a_{ij} = 1$ when i and j are connected, otherwise 0. The graph is not a priori living in Euclidean space. *But it might be! Clearly it could at worst be embedded in R^{d-1}, but what's the minimal dimension? Can you figure that out by listening to the eigenvalues?*

Combinatorial graphs

- We could debate what we mean by embedding, and our decision is *not* the one that is most usual in graph theory, which is just whether the graph can be drawn in the plane or not.

Combinatorial graphs

- Our choice is whether the graph is a subset of a regular d-dimensional lattice (possibly including some diagonals).
- I’ll refer to the minimal d for which a graph G is isomorphic to a subset of a regular lattice as its “embedding dimension.”

More vs less efficient embeddings of a 1D graph.

Connecting the spectrum of a graph and its embedding dimension

We can mimic the argument for the Kröger-type bounds by considering test functions of the form exp(ip•x), where x is restricted to integer vectors.

Is there a Parseval relation?   Yes
Is there a nice expression for the energy form? Yes

Fourier transform

The Fourier transform has various normalizations; the one I'll use is

$$\hat{f}(\mathbf{p}) := \frac{1}{(2\pi)^{d/2}} \int f(\mathbf{x}) e^{-\mathbf{p}\cdot\mathbf{x}} dp$$

Fourier transform

Its completeness, or Parseval relation (actually known to Fourier) reads:

$$\int |\hat{f}(\mathbf{p})|^2 d\mathbf{p} = \int |f(\mathbf{x})|^2 d\mathbf{x}$$

Connecting the spectrum of a graph and its embedding dimension

$$\overline{\lambda_k} \leq 2m\kappa \left(1 - \mathrm{sinc}(\kappa^{1/d}\pi)\right),$$

where $\kappa := \frac{k}{n}$, and $sinc(x) := \frac{sin(x)}{x}$.

Connecting the spectrum of a graph and its embedding dimension

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With Taylor,

$$\overline{\lambda_k} \leq \frac{\pi^2 m}{3}\kappa^{\frac{2}{d}} - \dots$$

$$M = [-\pi, \pi]^\nu$$

$$M_0 := [-a\pi, a\pi]^\nu$$

$$f_{\mathbf{x}} = \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$\hat{\phi}(\mathbf{x}) := \sum_{\mathbf{k} \in G} \mathrm{e}^{-i\mathbf{k} \cdot \mathbf{x}}\, \phi_{\mathbf{k}}, \qquad \phi_{\mathbf{k}} = \frac{1}{(2\pi)^\nu} \int_{[-\pi,\pi]^\nu} \mathrm{e}^{i\mathbf{k} \cdot \mathbf{x}}\, \hat{\phi}(\mathbf{x})$$

$$\left\langle H\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}}, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}}\right\rangle = \frac{1}{2}\sum_{\mathbf{k}\in G}\sum_{\mathbf{p}\sim\mathbf{k}} |\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} - \mathrm{e}^{i\mathbf{p}\cdot\mathbf{z}}\,|^2$$

$$|\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} - \mathrm{e}^{i\mathbf{p}\cdot\mathbf{z}}\,|^2 \text{ simplifies to } |e^{\pm i z_q} - 1|^2 = 4\sin^2\left(\tfrac{z_q}{2}\right)$$

$$A_0 = 2(a\pi - \sin(a\pi))(2a\pi)^{\nu-1}\sum_{\mathbf{k}\in G} d_k = (2a\pi)^{\nu}2\left(1 - \frac{\sin(a\pi)}{a\pi}\right)\mathcal{E}$$

$$\int_{[-\pi,\pi]^{\nu}} |\left\langle \mathrm{e}^{i\mathbf{k}\dot{\mathbf{x}}}, \phi_j\right\rangle|^2 = (2\pi)^{\nu}\|\phi_j\|^2 = (2\pi)^{\nu}$$

Meanwhile, on the left,

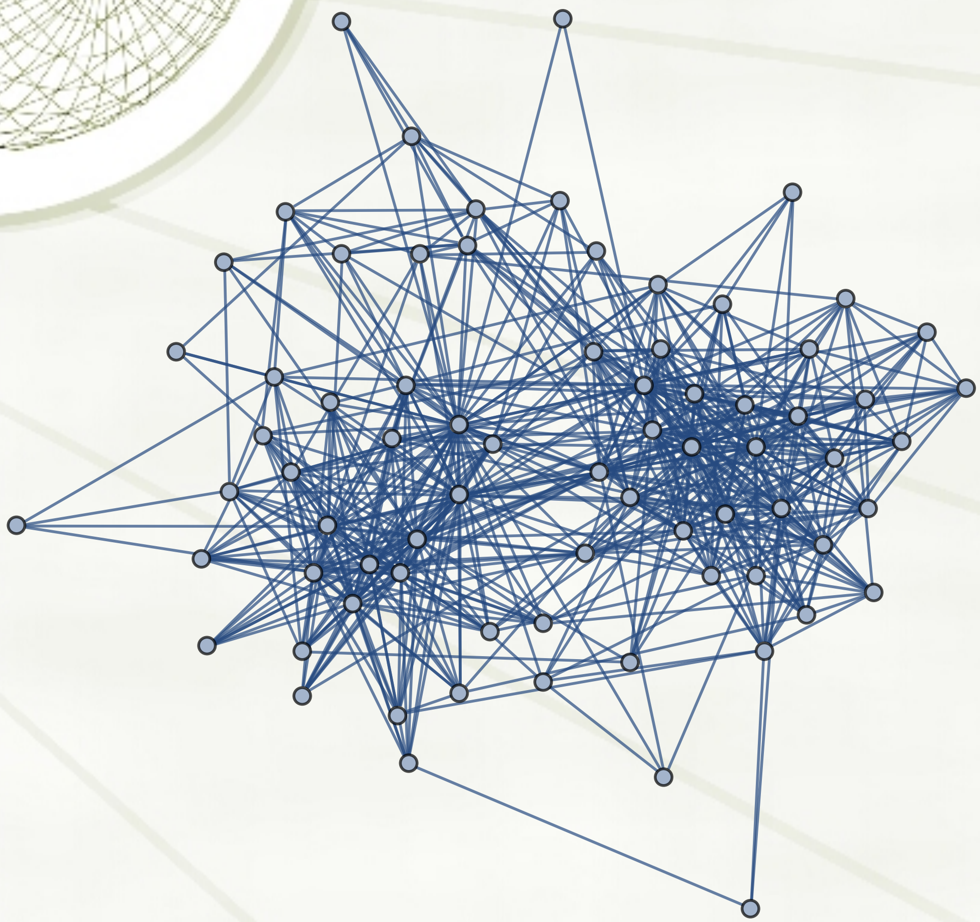
$$n(2a\pi)^\nu - k(2\pi)^\nu \geq 0,$$

so $a^\nu \to k/n$

Giving

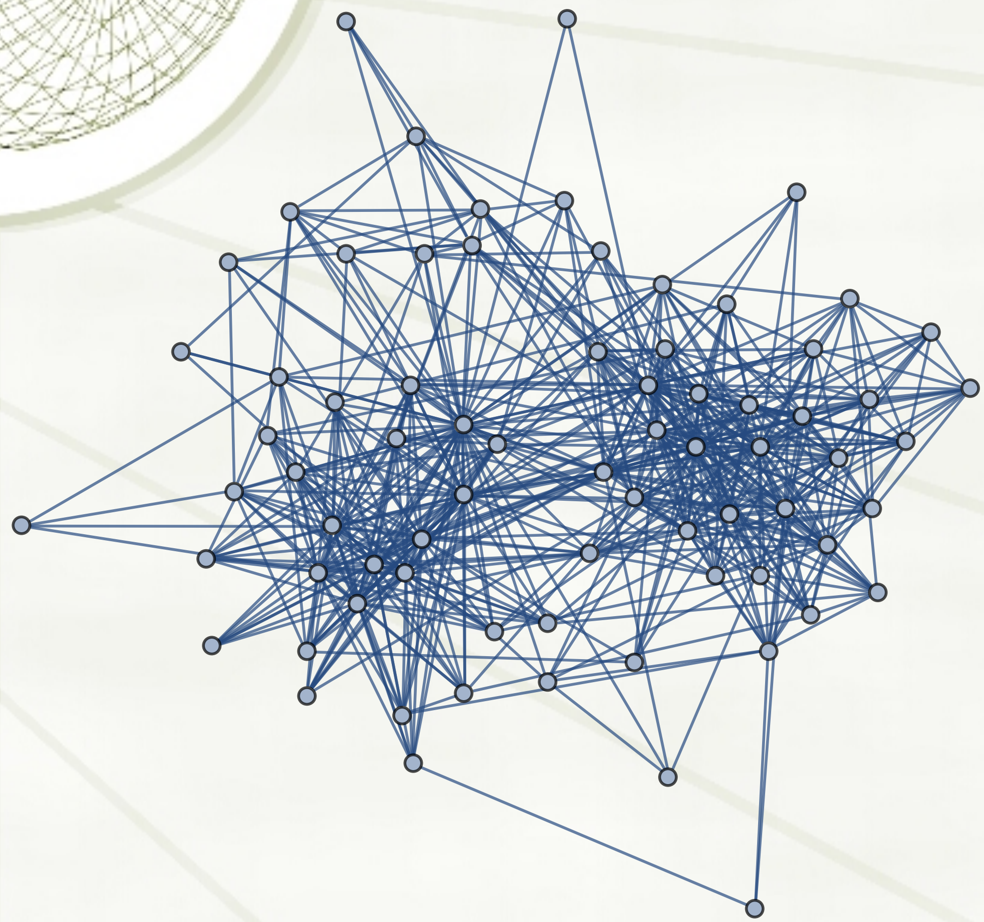
$$\sum_{j=1}^{k-1} \lambda_j \leq 2\mathcal{E}\left(1 - \frac{\sin((k/n)^{1/\nu}\pi)}{(k/n)^{1/\nu}\pi}\right)\frac{k}{n}$$

Dimension and complexity



This is the graph of wikipedia PDE pages. How many independent kinds of information (“dimensions”) are there?

Dimension and complexity



You might not see it visually, but the spectrum says that this is 3D!

Another interesting question:

Can you distinguish dimensions on different scales?

Variational bounds on graph spectra

Another way to apply the averaged variational principle to graphs is to let $\mathcal{M}$ be the set of pairs of vertices. The reason is that the complete graph has a *tight frame* of nontrivial eigenfunctions $b_{u,v}$ consisting of functions equal to 1 on one vertex, -1 on a second, and 0 everywhere else.

Variational bounds on graph spectra

Two facts are easily seen:

1. For vectors of mean 0 (orthogonal to **1**),

$$\sum_{u,v} | \langle b_{u,v}, f \rangle | = 2(n-1)\|f\|^2$$

2. $$\langle H b_{u,v}, b_{u,v} \rangle = d_u + d_v + 2a_{uv}$$

Variational bounds on graph spectra

From the averaged variational principle,

$$\sum_{j=1}^{k-1} \lambda_j \leq \frac{1}{2n} \sum_{\{u,v\} \in \mathfrak{M}_0} (d_u + d_v + 2a_{uv})$$

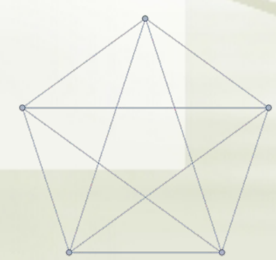
where $\mathfrak{M}_0$ is any set of pairs of vertices with cardinality nL.

Variational bounds on graph spectra

From the averaged variational principle,

$$\sum_{j=1}^{k-1} \lambda_j \leq \frac{1}{2n} \sum_{\{u,v\} \in \mathfrak{M}_0} (d_u + d_v + 2a_{uv})$$

The average eigenvalue knows how many vertices are “unfriendly” – with few connections to others. The inequality is an identity for star graphs, but also for complete graphs!



Variational bounds on graph spectra

Another set of test functions yields

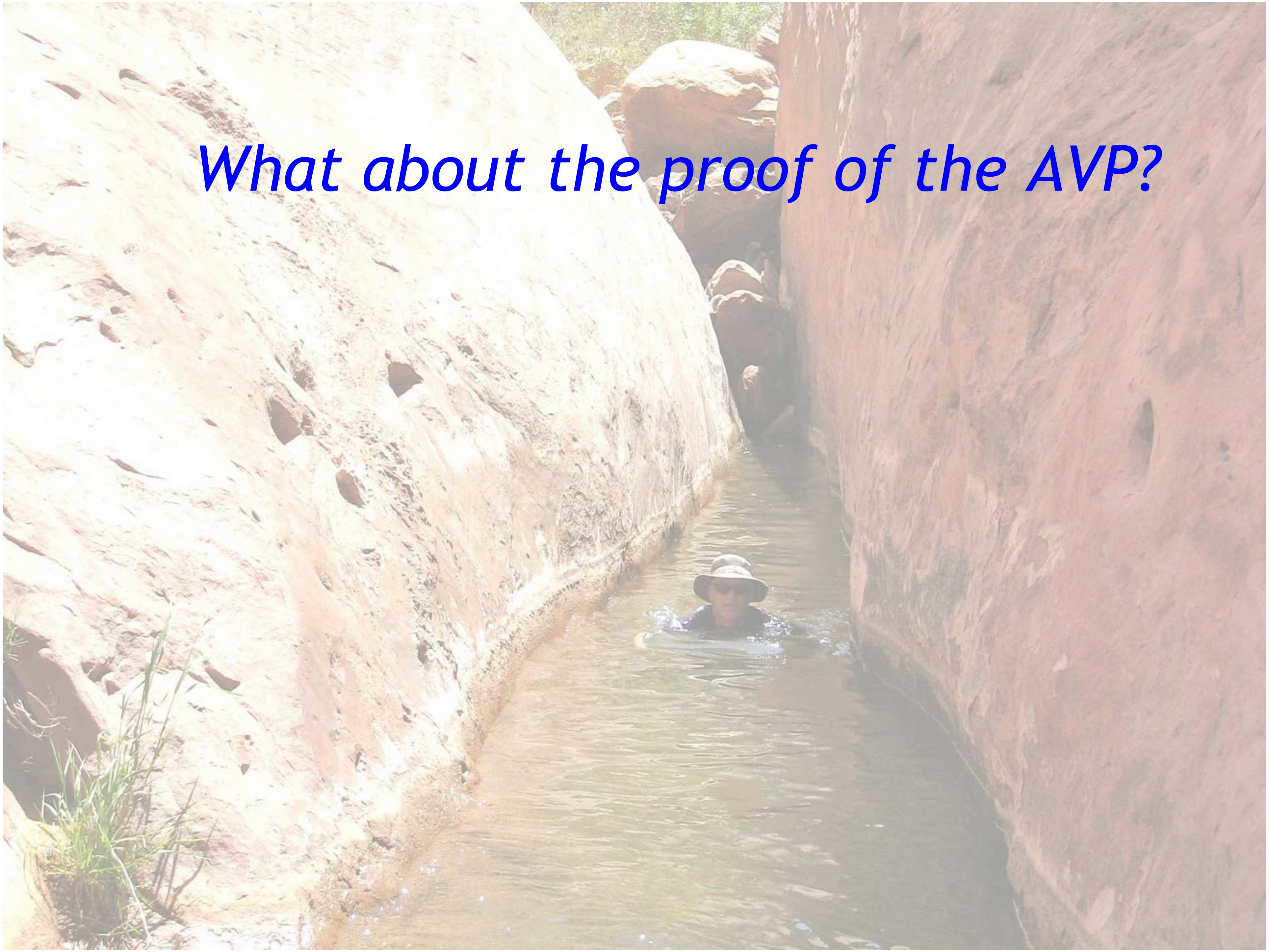
$$\sum_{i=1}^{L} \lambda_i \leq \frac{L}{L+1} \sum_{x=1}^{L+1} d_x + \frac{1}{L+1} \sum_{\substack{u=1 \\ v \neq u}}^{L+1} \sum_{v=1}^{L+1} a_{uv} \leq \sum_{j=n-L+1}^{n} \lambda_j$$

$$\sum_{i=1}^{L} \lambda_i \leq \frac{n-L+1}{n-L} \sum_{x=n-L+1}^{n} d_x + \frac{1}{n-L} \sum_{\substack{u=n-L+1 \\ v \neq u}}^{n} \sum_{v=n-L+1}^{n} a_{uv} \leq \sum_{j=n-L+1}^{n} \lambda_j.$$

These inequalities are likewise sharp for stars and complete graphs. They extend the old result of Fiedler, that

$$\lambda_1 \leq \frac{n}{n-1} \min_v d_v, \quad \frac{n}{n-1} \max_v d_v \leq \lambda_{n-1}$$

What about the proof of the AVP?



Proof

$$\mu_k\Big(\langle f,f\rangle - \langle P_{k-1}f, P_{k-1}f\rangle\Big) \le \langle Mf,f\rangle - \langle MP_{k-1}f, P_{k-1}f\rangle. \tag{3.1}$$

By integrating (3.1),

$$\begin{aligned}\mu_k \int_{\mathfrak{M}_0} (\langle f_z, f_z\rangle - \langle P_{k-1}f, P_{k-1}f_z\rangle)\, d\sigma \\ \le \int_{\mathfrak{M}_0} \langle Mf_z, f_z\rangle\, d\sigma - \int_{\mathfrak{M}_0} \langle MP_{k-1}f_z, P_{k-1}f_z\rangle\, d\sigma,\end{aligned} \tag{3.3}$$

Proof

or

$$\mu_k \int_{\mathfrak{M}_0} \left(\langle f_z, f_z \rangle - \sum_{j=0}^{k-1} |\langle f_z, \psi^{(j)} \rangle|^2 \right) d\sigma \tag{3.4}$$
$$\leq \int_{\mathfrak{M}_0} \langle M f_z, f_z \rangle \, d\sigma - \int_{\mathfrak{M}_0} \sum_{j=0}^{k-1} \mu_j |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma.$$

Since μ_k is larger than or equal to any weighted average of $\mu_1 \ldots \mu_{k-1}$, we add to (3.4) the inequality

$$-\mu_k \int_{\mathfrak{M} \setminus \mathfrak{M}_0} \left(\sum_{j=0}^{k-1} |\langle f_z, \psi^{(j)} \rangle|^2 \right) d\sigma \leq - \int_{\mathfrak{M} \setminus \mathfrak{M}_0} \sum_{j=0}^{k-1} \mu_j |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma, \tag{3.5}$$

and obtain the claim. □

How to use the averaged variational principle to get sharp results?

$$
\begin{aligned}
&\mu_k\Big(\int_{\mathfrak{M}_0}\cancel{\langle f_z, f_z\rangle d\sigma - \sum_{j=0}^{k-1}\int_{\mathfrak{M}}|\langle f_z, \psi^{(j)}\rangle}|^2\, d\sigma\Big)\\
&\leq \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.2)\\
&\int_{\mathfrak{M}_0}\langle H f_z, f_z\rangle d\sigma - \sum_{j=0}^{k-1}\mu_j\int_{\mathfrak{M}}|\langle f_z, \psi^{(j)}\rangle|^2\, d\sigma,
\end{aligned}
$$

How to use the averaged variational principle to get sharp results?

$$\sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma \leq \int_{\mathfrak{M}_0} \langle M f_z, f_z \rangle d\sigma.$$

How to use the averaged variational principle to get sharp results?

Ans: If $\mathfrak{M}_0$ is large enough that

$$\int_{\mathfrak{M}_0} \langle f_\zeta, f_\zeta \rangle \, d\sigma \geq \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma$$

then

$$\sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_\zeta, \psi^{(j)} \rangle|^2 \, d\sigma \leq \int_{\mathfrak{M}_0} \langle M f_\zeta, f_\zeta \rangle d\sigma$$

How to use the averaged variational principle to get sharp results?

If the test functions f_ζ and the measure space $\mathfrak{M}$ satisfy an abstract Parseval identity, then the theorem becomes a variational principle for sums of eigenvalues, as follows.

Corollary 1.1. *Under the assumptions of the Theorem, suppose further that* $\|f_\zeta\|^2 = C$ *is independent of* ζ*, and that for all* $\phi \in \mathcal{H}$*,* $\int_{\mathfrak{M}} |\langle \phi, f_\zeta \rangle|^2 \, d\sigma = A\|\phi\|^2$ *for a fixed* $A > 0$*. Then for any* $\mathfrak{M}_0 \subset \mathfrak{M}$ *such that* $|\mathfrak{M}_0| = k\frac{A}{C}$*,*

$$\frac{1}{k}\sum_{j=0}^{k-1} \mu_j \leq \frac{1}{|\mathfrak{M}_0|} \int_{\mathfrak{M}_0} \frac{Q_M(f_\zeta, f_\zeta)}{\|f_\zeta\|^2} d\sigma. \tag{8}$$


THE END