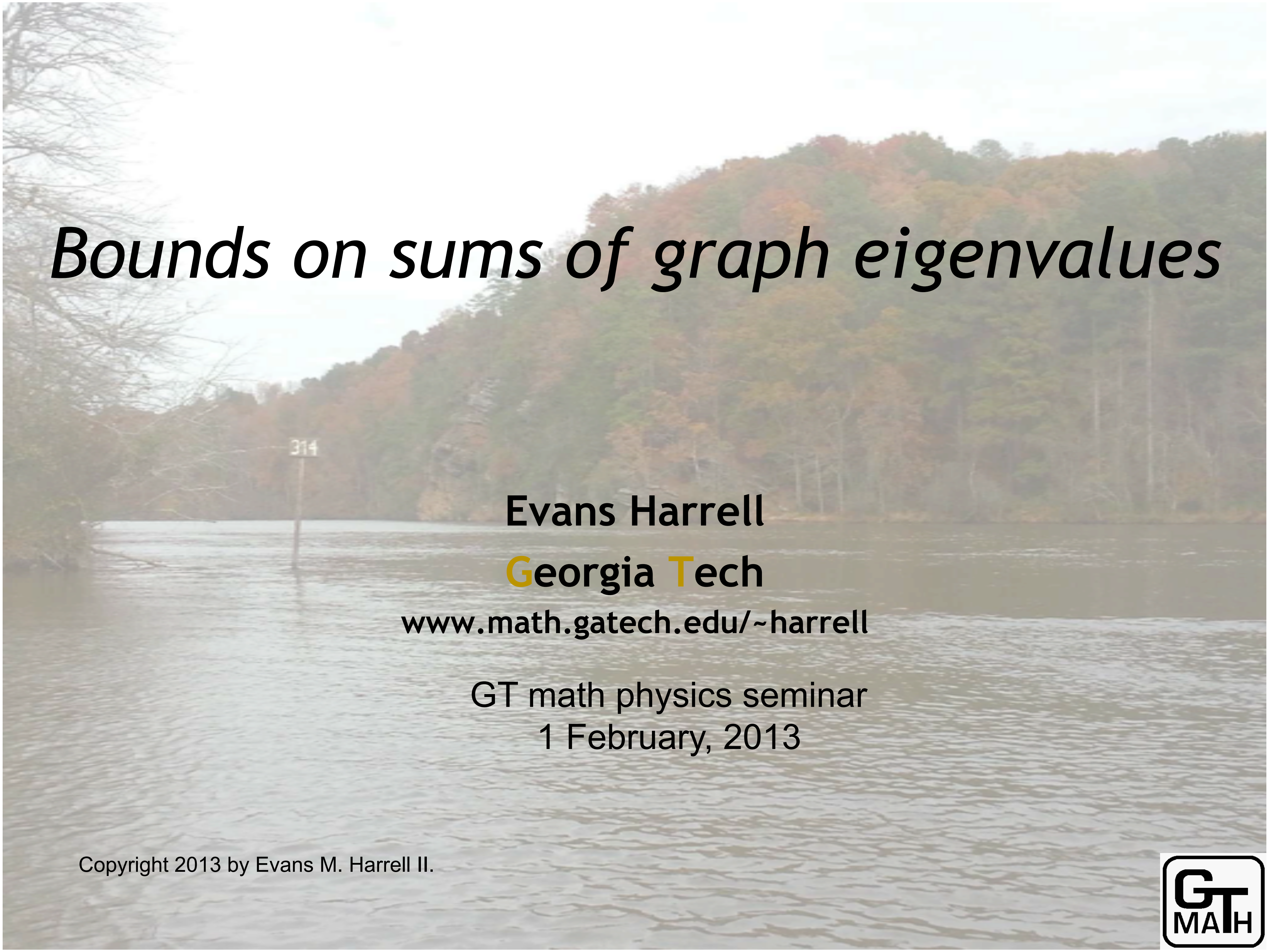


# *Bounds on sums of graph eigenvalues*

**Evans Harrell**

**Georgia Tech**

**www.math.gatech.edu/~harrell**

GT math physics seminar
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# *Abstract*


- I'll discuss two methods for finding bounds on sums of graph eigenvalues (variously for the Laplacian, the renormalized Laplacian, or the adjacency matrix). One of these relies on a Chebyshev-type estimate of the statistics of a subsample of an ordered sequence, and the other is an adaptation of a variational argument used by P. Kröger for Neumann Laplacians.  Some of the inequalities are sharp in suitable senses.
- This is ongoing work with J. Stubbe of ÉPFL.

## *The essential message of this seminar*

- It is well known that the largest and smallest eigenvalues, and some other spectral properties, such as determinants, satisfy simple inequalities and provide information about the structure of a graph.
- It will be shown that statistical properties of spectra (means, variance of samples) also satisfy simple inequalities and provide information about the structure of a graph.

## *Why focus on sums of eigenvalues?*
## $\lambda_0 + \ldots + \lambda_{k-1}$

- These are ground-state energies, in a weakly interacting approximation for matter obeying Fermi statistics.

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- They appear in "semiclassical" theorems related to Weyl limits and phase space, which have taken on a life of their own
  - Berezin-Li-Yau
  - Lieb-Thirring

*What would the semiclassical limit mean for a graph?*

# *A nanotutorial on graph spectra*

- A graph on n vertices is in 1-1 correspondence with an an n by n adjacency matrix A, with $a_{ij} = 1$ when i and j are connected, otherwise 0.

Generic assumptions: connected, not directed, finite, at most one edge between vertices, no self-connection...

# *A nanotutorial on graph spectra*

- A graph on n vertices is in 1-1 correspondence with an an n by n adjacency matrix A, with $a_{ij} = 1$ when i and j are connected, otherwise 0.
  - How is the structure of the graph reflected in the spectrum of A?
  - What sequences of numbers might be spectra of A?

# *A nanotutorial on graph spectra*

In[163]:= **TypEX0** = $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$;

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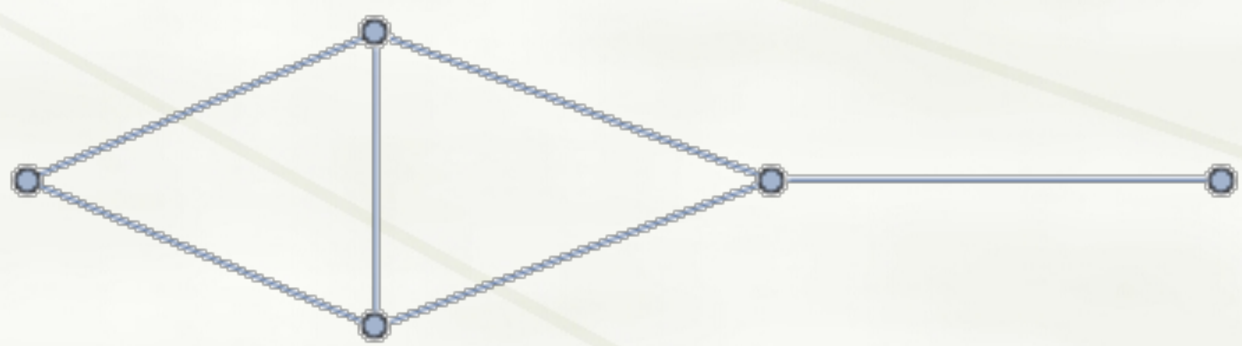

# *A nanotutorial on graph spectra*

- The *graph Laplacian* is a matrix that compares values of a function at a vertex with the average of its values at the neighbors.

$$H := -\Delta := \mathrm{Deg} - A, \text{ where}$$

$\mathrm{Deg} := \mathrm{diag}(d_v)$, $d_v :=$ # neighbors of v.

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- The *graph Laplacian* is a matrix that compares values of a function at a vertex with the average of its values at the neighbors.

$$H := -\Delta := \mathrm{Deg} - A, \text{ where}$$

Deg := diag($d_v$), $d_v$ := # neighbors of v.

- How is the structure of the graph reflected in the spectrum of $-\Delta$?
- What sequences of numbers might be spectra of $-\Delta$?

# *A nanotutorial on graph spectra*

- There is also a normalized graph Laplacian, favored by Fan Chung

$$C := Deg^{-\frac{1}{2}} H Deg^{-\frac{1}{2}}$$

# *A nanotutorial on graph spectra*

- There is also a normalized graph Laplacian, favored by Fan Chung.
- The spectra of the three operators are trivially related if the graph is regular (all degrees equal), but otherwise not.

## *The most basic spectral facts*

- The spectrum of A allows one to count "spanning subgraphs."
- It easily determines whether the graph has 2 colors. "bipartite"
- The max eigenvalue is ≤ the max degree.
- There is an interlacing theorem when an edge is added.

# *The most basic spectral facts*

- $H \geq 0$ and $H\,\mathbf{1} = 0\,\mathbf{1}$. (Like Neumann)
- Taking unions of disjoint edge sets,

$$H_{G_1 \cup G_2} = H_{G_1} + H_{G_2}$$

- This implies a relation between the spectra of a graph and of its edge complement, and various useful simple inequalities.
- The spectrum determines the number of spanning trees (classic thm of Kirchhoff)
- There is an interlacing theorem when an edge is added

# *The most basic spectral facts*

- For none of the operators on graphs is it known which precise sets of eigenvalues are feasible spectra.
- Examples of nonequivalent isospectral graphs are known (and not too tricky)
- Eigenfunctions can sometimes be supported on small subsets

# *Notation for the eigenvalues is not standardized!*

*Our notation for the eigenvalues of these three matrices is as follows:*

$$
\begin{aligned}
A: &\quad \alpha_0 > \alpha_1 \geq \ldots \alpha_{n-1} \\
H: &\quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \lambda_{n-1} \\
C: &\quad 0 = \mathfrak{c}_0 < \mathfrak{c}_1 < \mathfrak{c}_2 \leq \ldots \mathfrak{c}_{n-1} \leq 2.
\end{aligned}
$$

The indexing scheme ensures that in the case of a regular graph of degree $d$, $\lambda_k = d - \alpha_k = d\mathfrak{c}_k$ for each $k$.

# *Two good philosophies for understanding spectra*

- Make variational estimates.
- Exploit algebraic properties of the operator.

# *Variational bounds on graph spectra*

# *Variational bounds on graph spectra*

- By adapting Pawel Kröger's variational argument for the Neumann counterpart to Berezin-Li-Yau, using the basis for the complete graph in place of exp(i x•z), and averaging over edges instead of integrating over z, we get upper bounds on $\sum_{\ell}^{L} \lambda_\ell$ in terms of L and the degrees, and corresponding lower bounds on sums from L+1 to n-1.

## *An abstract version of Kröger's inequality*

**Lemma 5** *Consider a self-adjoint operator $H$ with ordered, entirely discrete spectrum $-\infty < \lambda_0 \le \lambda_1 \le \dots$ and corresponding normalized eigenvectors $\{\phi_k\}$. Let $f_z$ be a family of vectors in $D(H)$ indexed by a variable $z$ ranging over a measure space $(M, \Sigma, \mu)$. Suppose that $M_0$ is a subset of $M$. Then:*

$$\lambda_k\left(\int_{M_0}\langle f_z, f_z\rangle\, d\mu - \int_M \sum_{j=0}^{k-1} |\langle f_z, \phi_j\rangle|^2\, d\mu\right) \le \int_{M_0}\langle H f_z, f_z\rangle d\mu - \int_M \sum_{j=0}^{k-1}\lambda_j |\langle f_z, \phi_j\rangle|^2\, d\mu, \tag{2.14}$$

*provided that the integrals converge.*

## *Proof of Kröger's inequality*

By the variational principle,

$$\lambda_k\Big(\langle f, f\rangle - \langle P_{k-1}f, P_{k-1}f\rangle\Big) \leq \langle Hf, f\rangle - \langle HP_{k-1}f, P_{k-1}f\rangle. \qquad (2.13)$$

*Proof.* By integrating (2.13),

$$\lambda_k \int_{M_0} (\langle f_z, f_z\rangle - \langle P_{k-1} f, P_{k-1} f_z\rangle)\, d\mu \leq \int_{M_0} \langle H f_z, f_z\rangle\, d\mu - \int_{M_0} \langle H P_{k-1} f_z, P_{k-1} f_z\rangle\, d\mu, \tag{2.15}$$

or

$$\lambda_k \int_{M_0} \left(\langle f_z, f_z\rangle - \sum_{j=0}^{k-1} |\langle f_z, \phi_j\rangle|^2\right) d\mu \leq \int_{M_0} \langle H f_z, f_z\rangle\, d\mu - \int_{M_0} \sum_{j=0}^{k-1} \lambda_j |\langle f_z, \phi_j\rangle|^2\, d\mu. \tag{2.16}$$

Since $\lambda_k$ is larger than or equal to any weighted average of $\lambda_1 \ldots \lambda_{k-1}$, we add to (2.16) the inequality

$$-\lambda_k \int_{M\setminus M_0} \left(\sum_{j=0}^{k-1} |\langle f_z, \phi_j\rangle|^2\right) d\mu \leq - \int_{M\setminus M_0} \sum_{j=0}^{k-1} \lambda_j |\langle f_z, \phi_j\rangle|^2\, d\mu, \tag{2.17}$$

and obtain the claim. □

# *Kröger's inequality is used in an odd way*

Although it looks as though we want an upper bound on $\lambda_k$, Kröger instead arranged that the left side be ≥ 0, and focused on the implication that

$$\int_M \sum_{j=0}^{k-1} \lambda_j |\langle f_z, \phi_j \rangle|^2 \, d\mu \le A_0 := \int_{M_0} \langle H f_z, f_z \rangle d\mu$$

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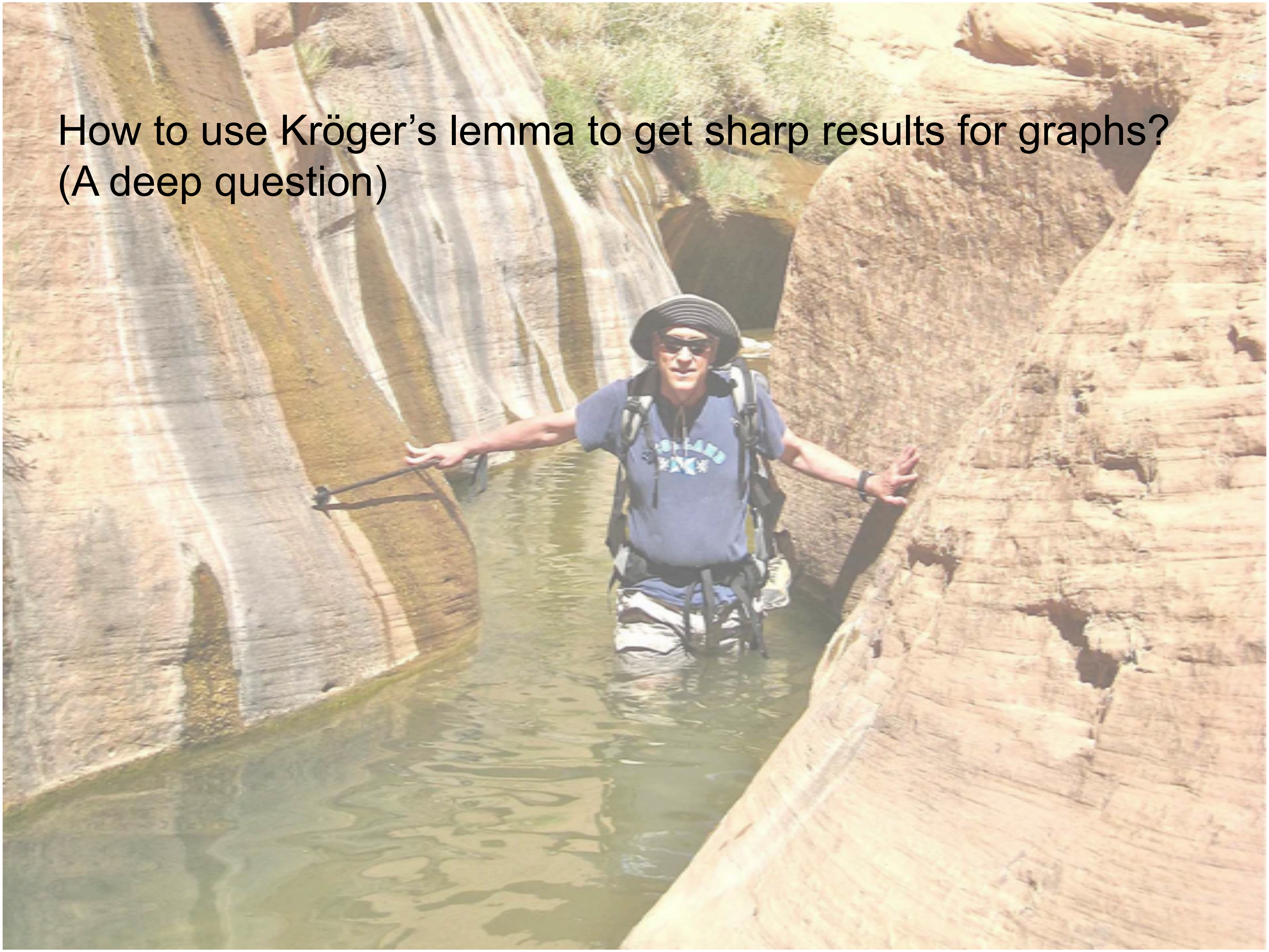
We want the "bigger" average to simplify the Fourier coefficients, while choosing the "smaller" average sufficiently large to arrange that $\lambda_k \left( \int_{M_0} \langle f_z, f_z \rangle \, d\mu - \int_M \sum_{j=0}^{k-1} |\langle f_z, \phi_j \rangle|^2 \, d\mu \right) \geq 0$.

How to use Kröger’s lemma to get sharp results for graphs?
(A deep question)



**Corollary 6** *Suppose that $G$ is a finite subgraph of $\Omega^\nu$. Then for $k \geq 2$ the eigenvalues of the graph Laplacian $H_G$ satisfy*

$$\sum_{j=1}^{k-1} \lambda_j \leq 2\mathcal{E}\left(1 - \frac{\sin((k/n)^{1/\nu}\pi)}{(k/n)^{1/\nu}\pi}\right)\frac{k}{n}, \qquad (2.17)$$

*where $\mathcal{E}$ denotes the number of edges of $G$.*

**Remark 2.2** *In particular, it is true independently of dimension that*

$$\sum_{j=1}^{k-1} \lambda_j \leq \frac{2\mathcal{E}k}{n},$$

*which becomes a standard equality when $k = n$. In the complementary situation where $k << n$ the upper bound is*

$$\sim \frac{\pi^2\mathcal{E}}{3}\left(\frac{k}{n}\right)^{1+\frac{2}{\nu}},$$

*which has the form of the Weyl law for Laplacians on domains $\Omega \subset \mathbb{R}^\nu$.*

$$M = [-\pi, \pi]^\nu$$

$$M_0 := [-a\pi, a\pi]^\nu$$

$$f_{\mathbf{x}} = \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$\hat{\phi}(\mathbf{x}) := \sum_{\mathbf{k} \in G} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}, \qquad \phi_{\mathbf{k}} = \frac{1}{(2\pi)^\nu} \int_{[-\pi,\pi]^\nu} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \hat{\phi}(\mathbf{x})$$

$$\left\langle H\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}}, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} \right\rangle = \frac{1}{2} \sum_{\mathbf{k}\in G} \sum_{\mathbf{p}\sim\mathbf{k}} |\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} - \mathrm{e}^{i\mathbf{p}\cdot\mathbf{z}}\,|^2$$

$$\left\langle H\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}}, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} \right\rangle = \frac{1}{2} \sum_{\mathbf{k}\in G} \sum_{\mathbf{p}\sim\mathbf{k}} |\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} - \mathrm{e}^{i\mathbf{p}\cdot\mathbf{z}}\,|^2$$

$|\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} - \mathrm{e}^{i\mathbf{p}\cdot\mathbf{z}}\,|^2$ simplifies to $|e^{\pm i z_q} - 1|^2 = 4\sin^2\left(\frac{z_q}{2}\right)$

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$$A_0 = 2(a\pi - \sin(a\pi))(2a\pi)^{\nu-1} \sum_{\mathbf{k}\in G} d_k = (2a\pi)^{\nu} 2 \left(1 - \frac{\sin(a\pi)}{a\pi}\right) \mathcal{E}$$

$$\left\langle H\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}}, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} \right\rangle = \frac{1}{2}\sum_{\mathbf{k}\in G}\sum_{\mathbf{p}\sim\mathbf{k}} |\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} - \mathrm{e}^{i\mathbf{p}\cdot\mathbf{z}}\,|^2$$

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$$\int_{[-\pi,\pi]^{\nu}} |\left\langle \mathrm{e}^{i\mathbf{k}\dot{\mathbf{x}}}, \phi_j \right\rangle|^2 = (2\pi)^{\nu}\|\phi_j\|^2 = (2\pi)^{\nu}$$

Meanwhile, on the left we require that

$$n(2a\pi)^{\nu} - k(2\pi)^{\nu} \geq 0,$$

so $a^{\nu} \to k/n$

Giving

$$\sum_{j=1}^{k-1} \lambda_j \leq 2\mathcal{E}\left(1 - \frac{\sin((k/n)^{1/\nu}\pi)}{(k/n)^{1/\nu}\pi}\right)\frac{k}{n}$$

# *Variational bounds on graph spectra*

Another way to apply the Kröger Lemma to graphs is to let $\mathcal{M}$ be the set of pairs of vertices. The reason is that the complete graph has a superbasis of nontrivial eigenfunctions consisting of functions equal to 1 on one vertex, -1 on a second, and 0 everywhere else. Let these functions be $h_z$, where z is a vertex pair.

# *Variational bounds on graph spectra*

Two facts are easily seen:

1. For vectors of mean 0 (orthogonal to $\phi_0 = 1$),

$$\sum_{\text{all pairs}} |\langle h_{uv}, f\rangle|^2 = 2\|f\|^2(n-1).$$

2. $$\langle Hh_{uv}, h_{uv}\rangle = d_u + d_v + 2a_{uv}$$

# *Variational bounds on graph spectra*

It follows from Kröger's lemma that

$$\sum_{j \leq L} \lambda_j \leq \frac{1}{2n} \min_{\text{choices of nL pairs}} \sum_{uv} (d_u + d_v + 2a_{uv})$$

# *Variational bounds on graph spectra*

- Extensions to renormalized Laplacian

**Corollary 8** *Let $G$ be any finite graph on $n$ vertices, and let $M_0$ be any set of $p$ pairs of vertices $\{u, v\}$ with $\sum_{M_0} d_u + d_v \geq 2(k-1)\mathcal{E}$. Then the eigenvalues of the renormalized Laplacian $C_G$ satisfy*

$$\sum_{j=1}^{k-1} \mathfrak{c}_j \leq \frac{1}{2\mathcal{E}} \sum_{M_0} (2 + d_u + d_v).$$

# *Variational bounds on graph spectra*

- Variant for adjacency matrix?

# *Variational bounds on graph spectra*

- Variant for adjacency matrix?
  - Caution! Since the spectrum is not a fortiori positive, to get a useful inequality the coefficient of $\lambda_k$ in the Kröger lemma should be 0, not just $\geq 0$. That turns out to be possible to arrange, but we are debugging our arithmetic.

# *Variational bounds on graph spectra*

- Other inequalities arise from min-max and good choices of trial functions.

- For example, Fiedler showed in 1973 that for the graph Laplacian

  $(0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_{n-1} \leq n)$

$$\lambda_1 \leq \frac{n}{n-1} \min_k d_k, \quad \frac{n}{n-1} \max_k d_k \leq \lambda_{n-1}$$

# *Variational bounds on graph spectra*

Alternative for $\lambda_1 + \lambda_2$:

$$\lambda_1 + \lambda_2 \leq \frac{2E}{n-2} + \frac{n(n-3)}{(n-1)(n-2)} \min_k d_k$$

# *Variational bounds on graph spectra*

*For any* $L = 1, \ldots, n-1$ *we get*

$$\sum_{i=1}^{L} \lambda_i \leq \frac{L}{L+1} \sum_{i=1}^{L+1} d_i + \frac{1}{L+1} \sum_{\alpha=1}^{L+1} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{L+1} a_{\alpha\beta} \leq \sum_{i=N-L+1}^{N} \lambda_i$$

$$\sum_{i=1}^{L} \lambda_i \leq \frac{n-L+1}{n-L} \sum_{i=n-L+1}^{n} d_i + \frac{1}{n-L} \sum_{\alpha=n-L+1}^{n} \sum_{\substack{\beta=n-L+1 \\ \beta\neq\alpha}}^{n} a_{\alpha\beta} \leq \sum_{i=N-L+1}^{N} \lambda_i.$$

- where the degrees are in decreasing order. Optimal for the complete and star graphs

# *Variational bounds on graph spectra*

- Generalization of Fiedler:

*For any* $L = 1, \ldots, n-1$ *we get*

$$\sum_{i=1}^{L} \lambda_i \leq \frac{L}{L+1} \sum_{i=1}^{L+1} d_i + \frac{1}{L+1} \sum_{\substack{\alpha=1 \\ \beta \neq \alpha}}^{L+1} \sum_{\beta=1}^{L+1} A_{\alpha\beta} \leq \sum_{i=N-L+1}^{N} \lambda_i$$

$$\sum_{i=1}^{L} \lambda_i \leq \frac{n-L+1}{n-L} \sum_{i=n-L+1}^{n} d_i + \frac{1}{n-L} \sum_{\substack{\alpha=n-L+1 \\ \beta \neq \alpha}}^{n} \sum_{\beta=n-L+1}^{n} A_{\alpha\beta} \leq \sum_{i=N-L+1}^{N} \lambda_i.$$

# *Variational bounds on graph spectra*

- Use eigenvectors of

$$H_p := \left(\begin{array}{cccc|cccc} p & 0 & \dots & 0 & -1 & -1 & \dots & -1 \\ 0 & p & \dots & 0 & -1 & -1 & \dots & -1 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & p & -1 & -1 & \dots & -1 \\ \hline -1 & -1 & \dots & -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & \dots & -1 & -1 & n-1 & & -1 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ -1 & -1 & \dots & -1 & -1 & -1 & \dots & n-1 \end{array}\right). \tag{2.1}$$

**Remark 1** *In particular, $H_1$ is the Laplacian of a star graph, while $H_{n-1}$ is the Laplacian of a complete graph. For future purposes we observe that*

$$tr(H_p) = p(2n-p-1), \quad tr(H_p^2) = p(n^2+pn-p^2-p) \tag{2.2}$$

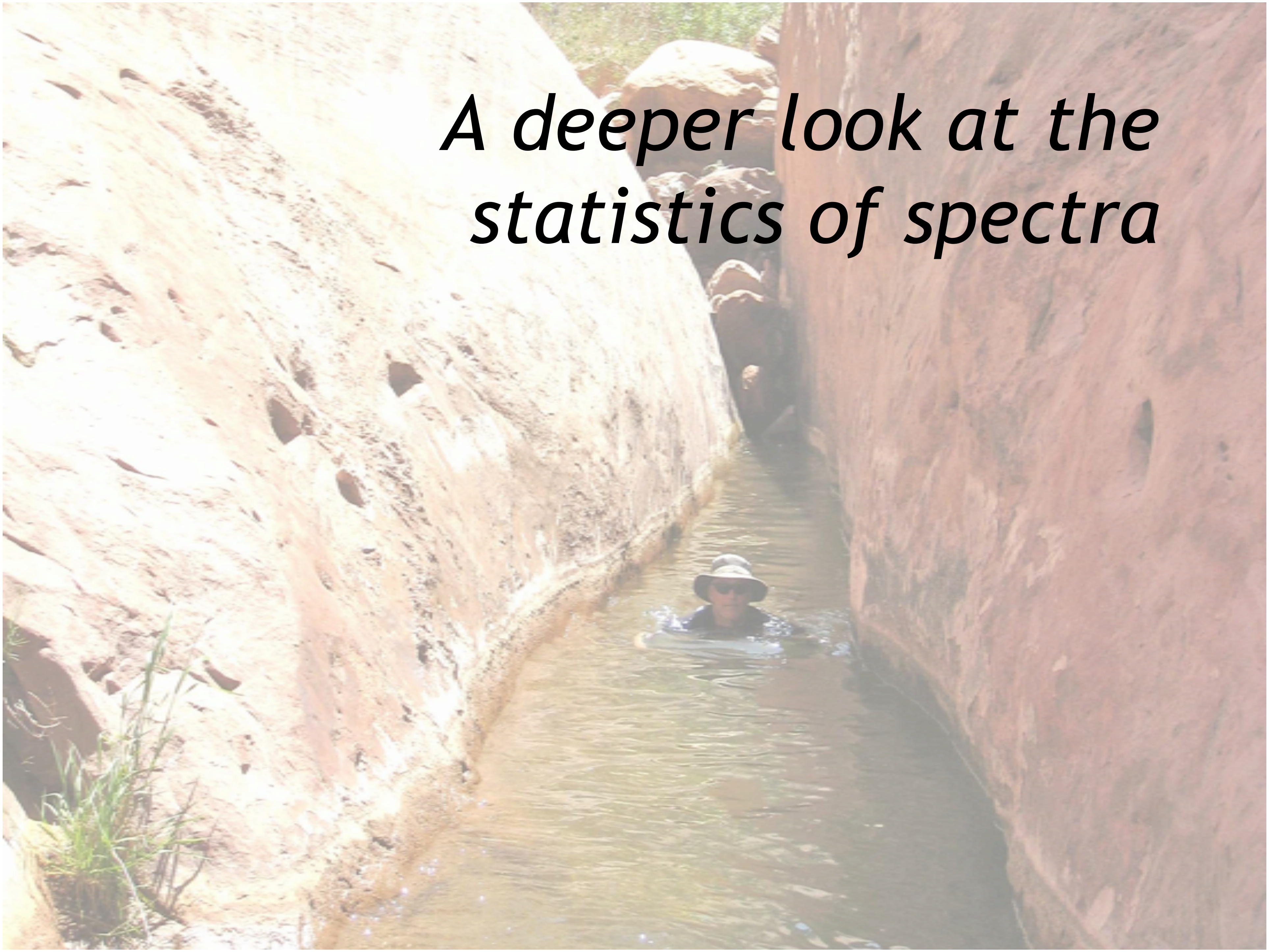
# *Variational bounds on graph spectra*

**Proposition 2.1** *(Spectral analysis of $H_p$.) Let*

$$\overline{\mathbf{e}}_k := \frac{1}{\sqrt{k(k+1)}}\Big(k\mathbf{e}_{k+1} - \sum_{j=1}^{k} \mathbf{e}_j\Big), \tag{2.3}$$

*where $\mathbf{e}_j, j = 1\ldots, n$ denote the canonical orthonormal basis vectors of $\mathbb{R}^n$. Then $\{\overline{\mathbf{e}}_k, k = 0\ldots, n-1\}$ is an orthonormal basis of $\mathbb{R}^n$. For each $k = 1, \ldots, n-p-1$, $\overline{\mathbf{e}}_k$ is an eigenvector of $H_p$ with corresponding eigenvalue $p$, and for each $k = n-p, \ldots, n-1$, $\overline{\mathbf{e}}_k$ is an eigenvector of $H_p$ with corresponding eigenvalue $n$.*

# *A deeper look at the statistics of spectra*



# Pafnuty Chebyshev

From Wikipedia, the free encyclopedia

*"Chebyshev" redirects here. For other uses, see Chebyshev (disambiguation).*

**Pafnuty Lvovich Chebyshev** (Russian: **Пафну́тий Льво́вич Чебышёв**, IPA: [pɐfˈnutʲɪj ˈlʲvovʲɪtɕ tɕɪbɨˈʂof]) (May 16 [O.S. May 4] 1821 – December 8 [O.S. November 26] 1894)[1] was a Russian mathematician. His name can be alternatively transliterated as **Chebychev** (English transliteration), **Chebysheff** (English), **Chebyshov** (English), **Tchebychev** (French) or **Tchebycheff** (French), or **Tschebyschev** (German) or **Tschebyscheff** (German) or **Tschebyschow** (German).

**Pafnuty Chebyshev**

Pafnuty Lvovich Chebyshev

| | |
|---|---|
| **Born** | May 16, 1821<br>Borovsk, Kaluga, Russian Empire |
| **Died** | December 8, 1894 (aged 73)<br>St Petersburg, Russian Empire |
| **Nationality** | Russian |
| **Fields** | Mathematician |
| **Institutions** | St Petersburg University |
| **Alma mater** | Moscow University |
| **Doctoral advisor** | Nikolai Brashman |
| **Doctoral students** | Dmitry Grave<br>Aleksandr Korkin<br>Aleksandr Lyapunov<br>Andrey Markov<br>Vladimir Andreevich Markov<br>Konstantin Posse |
| **Known for** | Mechanics and analytical geometry |
| **Notable awards** | Demidov Prize (1849) |

**Contents** [hide]



## Biography [edit]

One of nine children, he was born in the central Russian village of Okatovo near Borovsk, to Agrafena Ivanova Pozniakova and Lev Pavlovich Chebyshev. His father fought as an officer against Napoleon's invading army.

He was originally home-schooled by his mother and his cousin Avdotia Kvintillianova Soukhareva. He learned French early in life, which later helped him communicate with other mathematicians. A stunted leg prevented him from playing with other children, leading him to concentrate on studying instead.

Later he studied at Moscow University obtaining his degree in 1841.

He was a student of Nikolai Brashman. His own most illustrious student was Andrey Markov, although Alexandr Lyapunov is also famous for the method that bears his name.

Chebyshev died in St Petersburg on 26 November 1894.

## Mathematical contributions [edit]

Chebyshev is known for his work in the field of probability, statistics and number theory. Chebyshev's inequality says that if $X$ is a random variable with standard deviation σ, the probability that the outcome of $X$ is no less than $a\sigma$ away from its mean is no more than $1/a^2$:

$$\Pr(|X-\mathbf{E}(X)| \geq a\,\sigma) \leq \frac{1}{a^2}.$$

Chebyshev's inequality is used to prove the weak law of large numbers.

The Bertrand–Chebyshev theorem (1845|1850) states that for any $n > 1$, there exists a prime number $p$ such that $n < p < 2n$. It is a consequence of Chebyshev inequalities for the number $\pi(n)$ of prime numbers less than $n$, which state that $\pi(n)$ is of the order of $n/\log(n)$. A more precise form is given by the celebrated prime number theorem: the *quotient* of the two expressions approaches 1 as $n$ tends to infinity.

## Legacy [edit]

Chebyshev is considered a founding father of Russian mathematics. Among his well-known students were the prolific mathematicians Dmitry Grave, Aleksandr Korkin, Aleksandr Lyapunov and Andrey Markov. According to the Mathematics Genealogy Project, Chebyshev has 7,483 mathematical

# *Пафнутий Львович Чебышёв*

1. Inequalities involving means and standard deviations of ordered sequences.  References: Hardy-Littlewood-Pólya, Mitrinovic.

# *Riesz means*

- The counting function,

  $N(z) := \#(\lambda_k \leq z)$
- Integrals of the counting function, known as *Riesz means*

$$R_p(z) := \sum_j (z - \lambda_j)_+^p$$

- Chandrasekharan and Minakshisundaram, 1952; Safarov, Laptev, Weidl, ...

**Lemma 2.1** *Given any finite sequence* $\Sigma = \{x_1, \ldots, x_n\}$, *let* $\overline{x} := \frac{1}{n}\sum_n x_n$, $\overline{x^2} := \frac{1}{n}\sum_n x_n^2$, *and* $\sigma^2 := \overline{x^2} - \overline{x}^2$, *Then for each real number* $z$,

$$\begin{aligned}&\sum_{x_j \in J} z^2(\overline{x} - x_j) - z(\overline{x^2} - x_j^2) + x_j\overline{x^2} - x_j^2\overline{x} \\ &= \frac{1}{n}\sum_{x_j \in J}\sum_{x_k \in J^c}(z - x_j)(z - x_k)(x_k - x_j).\end{aligned} \tag{2.2}$$

*As a consequence,*

$$(z - \overline{x})\, R_2(z) \le (z - \overline{x})^2\, R_1(z) + \sigma^2 R_1(z), \tag{2.3}$$

*and*

$$\frac{R_2(z)}{(z - \overline{x})^2 + \sigma^2}$$

*is a nondecreasing function of* $z$.

If the sequence happens to be the spectrum of a self-adjoint matrix, then

$$\sum_{\lambda_j \in J} z^2(\operatorname{tr}(H) - n\lambda_j) - z(\operatorname{tr}(H^2) - n\lambda_j^2) + \lambda_j \operatorname{tr}(H^2) - \lambda_j^2 \operatorname{tr}(H)$$

$$= \sum_{\lambda_j \in J} \sum_{\lambda_k \in J^c} (z - \lambda_j)(z - \lambda_k)(\lambda_k - \lambda_j).$$

# *How can a general identity give information about graphs?*

# *How can a general identity give information about graphs?*

$$tr(H) = \sum_v d_v = 2\mathcal{E}$$

$$tr(H^2) = \sum_v (d_v^2 + d_v) = 2(\mathcal{E} + \zeta)$$

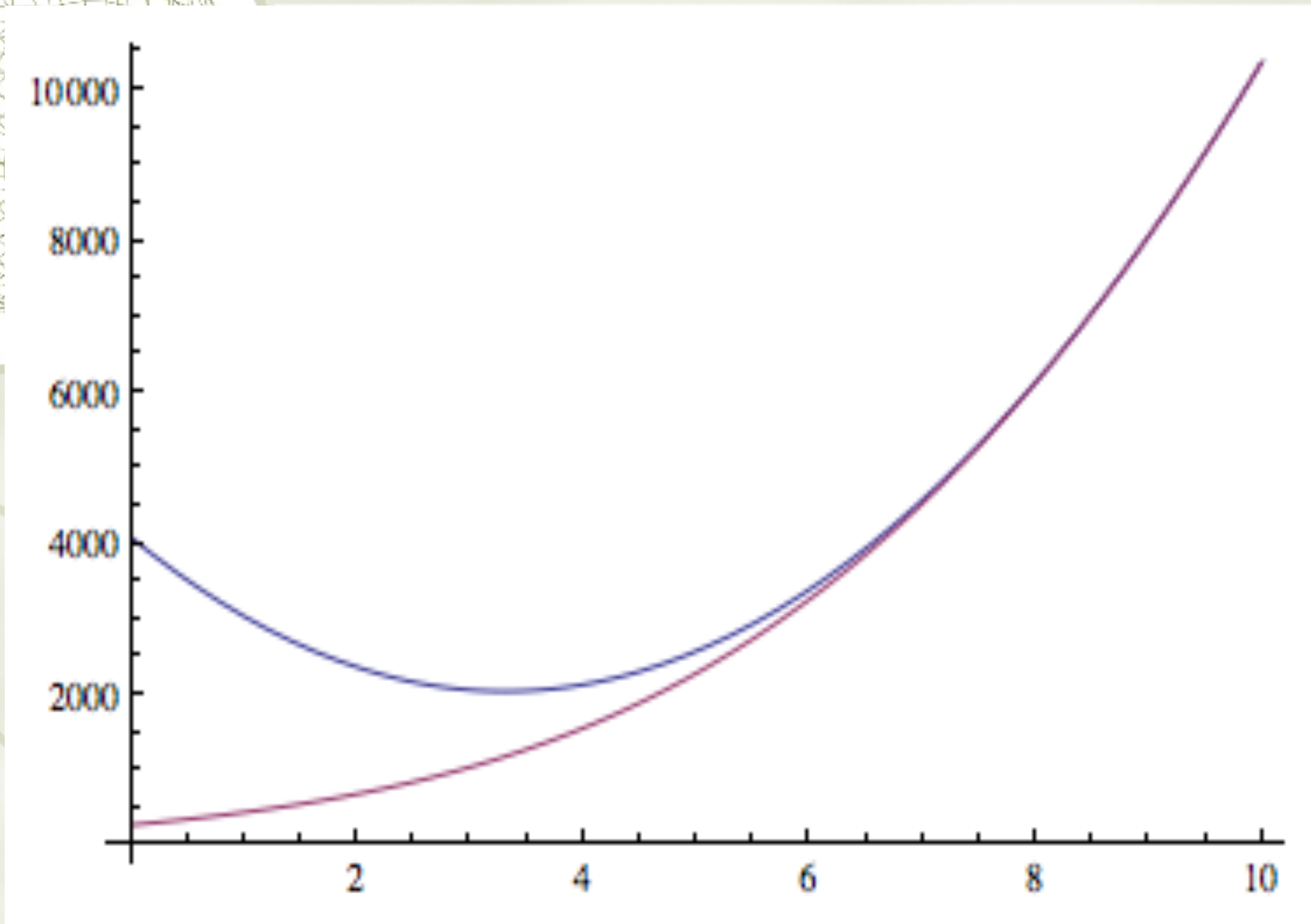
$$tr(H^3) = \sum_u (d_v^3 + 3d_v^2) - 6T$$

# *How can a general identity give information about graphs?*

$$\frac{R_2(z)}{z^2 - 2\mathcal{E}z + 2\zeta}$$

is a nondecreasing function of $z$.

This is sharp for complete graphs, and always has the limit 1, attained already for $z \geq \lambda_{n-1}$.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# *An analogue of Lieb-Thirring*

- Consider the operator s Deg – A, which interpolates between -A and H as s goes from 0 to 1. Then (writing D for Deg)

$$-\frac{d}{ds}\sum(z-\lambda_j)_+^3 \leq 3(z^2\,\mathrm{tr}(D) - 2zs\,\mathrm{tr}(D^2) + s^2\mathrm{tr}(D^3) + \mathrm{tr}(D^2)).$$

# *An analogue of Lieb-Thirring*

- When integrated,

$$\mathrm{tr}((z+A)_+^3) - \mathrm{tr}((z-H)_+^3) \leq 3z^2\,\mathrm{tr}(D) - 3z\,\mathrm{tr}(D^2) + \mathrm{tr}(D^3) + 3\mathrm{tr}(D^2)$$

i.e.,

$$\mathrm{tr}((z+A)_+^3) - \mathrm{tr}((z-H)_+^3) \leq \mathrm{tr}((z+A)^3) - \mathrm{tr}((z-H)^3)$$


THE END