On sums of eigenvalues of elliptic operators on homogeneous spaces

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Before we set out, ...

Happy 265th, Laplace!

Abstract

- Using a new variational technique for sums of eigenvalues, where orthogonalization is replaced by averaging, we derive sharp upper bounds on sums of eigenvalues for a wide category of elliptic operators on homogeneous spaces. Among the operators we can treat are Laplace-Beltrami-Schr\"odinger operators, the Witten Laplacian, and the operator of vibrations of inhomogeneous membranes. When the operator is defined on a domain with a boundary, Neumann conditions are imposed, in the weak sense. This is joint work with J. Stubbe of EPFL and A.h El Soufi and S. Ilias, Univ. de Tours.

- Work in progress; for a previous related article: arXiv: 1308.5340

Spectra, geometry, and dimensionality

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- Harrell-Hermi for $\overline{\lambda_k} := \frac{1}{k}\sum_{j=1}^{k}\lambda_j$

$$\frac{\overline{\lambda_k}}{\overline{\lambda_j}} \leq \frac{4+d}{2+d}\left(\frac{k}{j}\right)^{2/d}$$

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$$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}$$

- Lieb Thirring for Schrödinger operators with negative spectrum,

$$\sum_{\lambda_j<0} |\lambda_j|^p \leq L_{p,d} \int |V(x)|^{p+d/2}$$

Variational bounds on graph spectra

- In 1992 Pawel Kröger found a variational argument for the Neumann counterpart to Berezin-Li-Yau, i.e. a Weyl-sharp upper bounds on sums of the eigenvalues of the Neumann Laplacian.

- BLY: $$\sum_{j=1}^{k} \lambda_j \geq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}$$
- Kröger: $$\sum_{j=0}^{k-1} \mu_j \leq \frac{d}{d+2} \frac{4\pi^2 k^{1+2/d}}{(C_d|\Omega|)^{2/d}}$$

A new tool: an averaged variational principle for sums

An averaged variational principle for sums

Theorem 3.1 *Consider a self-adjoint operator M on a Hilbert space $\mathcal{H}$, with ordered, entirely discrete spectrum $-\infty < \mu_0 \le \mu_1 \le \dots$ and corresponding normalized eigenvectors $\{\psi^{(\ell)}\}$. Let f_z be a family of vectors in $\mathcal{Q}(M)$ indexed by a variable z ranging over a measure space $(\mathfrak{M}, \Sigma, \sigma)$. Suppose that $\mathfrak{M}_0$ is a subset of $\mathfrak{M}$. Then for any eigenvalue μ_k of M,*

$$\mu_k\left(\int_{\mathfrak{M}_0} \langle f_z, f_z \rangle \, d\sigma - \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma \right)$$

$$\le \tag{3.2}$$

$$\int_{\mathfrak{M}_0} \langle M f_z, f_z \rangle d\sigma - \sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma,$$

provided that the integrals converge.

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$$\begin{aligned} &\mu_k\left(\int_{\mathfrak{M}_0} \langle f_z, f_z\rangle \, d\sigma - \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_z, \psi^{(j)}\rangle|^2 \, d\sigma\right) \\ &\leq \\ &\int_{\mathfrak{M}_0} \langle M f_z, f_z\rangle d\sigma - \sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_z, \psi^{(j)}\rangle|^2 \, d\sigma, \end{aligned} \tag{3.2}$$

provided that the integrals converge.

E. Harrell, J. Stubbe, preprint 2013

Proof

$$\mu_k\Big(\langle f,f\rangle - \langle P_{k-1}f, P_{k-1}f\rangle\Big) \leq \langle Mf,f\rangle - \langle MP_{k-1}f, P_{k-1}f\rangle. \qquad (3.1)$$

Proof

$$\mu_k\Big(\langle f,f\rangle - \langle P_{k-1}f, P_{k-1}f\rangle\Big) \le \langle Mf,f\rangle - \langle MP_{k-1}f, P_{k-1}f\rangle. \tag{3.1}$$

By integrating (3.1),

$$\begin{aligned} &\mu_k \int_{\mathfrak{M}_0} (\langle f_z, f_z\rangle - \langle P_{k-1}f, P_{k-1}f_z\rangle)\, d\sigma \\ &\quad \le \int_{\mathfrak{M}_0} \langle Mf_z, f_z\rangle\, d\sigma - \int_{\mathfrak{M}_0} \langle MP_{k-1}f_z, P_{k-1}f_z\rangle\, d\sigma, \end{aligned} \tag{3.3}$$

Proof

or

$$\mu_k \int_{\mathfrak{M}_0} \left(\langle f_z, f_z \rangle - \sum_{j=0}^{k-1} |\langle f_z, \psi^{(j)} \rangle|^2 \right) d\sigma \tag{3.4}$$
$$\leq \int_{\mathfrak{M}_0} \langle M f_z, f_z \rangle \, d\sigma - \int_{\mathfrak{M}_0} \sum_{j=0}^{k-1} \mu_j |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma.$$

Since μ_k is larger than or equal to any weighted average of $\mu_1 \ldots \mu_{k-1}$, we add to (3.4) the inequality

$$-\mu_k \int_{\mathfrak{M} \setminus \mathfrak{M}_0} \left(\sum_{j=0}^{k-1} |\langle f_z, \psi^{(j)} \rangle|^2 \right) d\sigma \leq - \int_{\mathfrak{M} \setminus \mathfrak{M}_0} \sum_{j=0}^{k-1} \mu_j |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma, \tag{3.5}$$

and obtain the claim. □

How to use the averaged variational principle to get sharp results?

$$\mu_k\left(\int_{\mathfrak{M}_0} \langle f_z, f_z \rangle \, d\sigma - \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma \right) \\ \leq \\ \int_{\mathfrak{M}_0} \langle H f_z, f_z \rangle d\sigma - \sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma, \tag{3.2}$$

How to use the averaged variational principle to get sharp results?

$$\mu_k\Big(\int_{\mathfrak{M}_0}\langle f_z, f_z\rangle\, d\sigma - \sum_{j=0}^{k-1}\int_{\mathfrak{M}}|\langle f_z, \psi^{(j)}\rangle|^2\, d\sigma\Big)$$
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How to use the averaged variational principle to get sharp results?

$$\sum_{j=0}^{k-1} \mu_j \int_{\mathfrak{M}} |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma \leq \int_{\mathfrak{M}_0} \langle M f_z, f_z \rangle d\sigma.$$

Our first use of the averaged variational principle:

Harrell-Stubbe 2013: Weyl-type upper bounds on sums of eigenvalues of graph Laplacians and related operators.

PDEs on homogeneous spaces

A homogeneous space is a manifold M with a continuous symmetry group of isomorphisms $M \rightarrow M$.

Canonical examples: R^d, S^d, H^d.

The weak form of PDEs with Neumann BC

We can find sharp upper bounds for sums of eigenvalues of expressions defined in a variational quadratic form as follows:

$$\mathcal{E}(\varphi) := \frac{\int_\Omega (|\nabla\varphi(\mathbf{x})|^2 + V(\mathbf{x})|\varphi(\mathbf{x})|^2) w(\mathbf{x}) e^{-2\rho(\mathbf{x})} dv_g}{\int_\Omega |\varphi(\mathbf{x})|^2 e^{-2\rho(\mathbf{x})} dv_g},$$

Where Ω is a domain in a homogeneous space, which has been conformally transformed in an arbitrary way. Weak Neumann conditions correspond to test functions in the restriction of $H_0^1(R^d)$ to Ω. (Evans and Edmunds)

Example: Recover Kröger's result

Choose $\rho = 0$, $w = 1$, $V = 0$, and use as test functions $f = e^{ip\bullet x}$. Take $\mathfrak{M} = R^d$. Then our theorem says that IF $\mathfrak{M}_0$ is sufficiently large that

$$\int_{\mathfrak{M}_0} \langle f_z, f_z \rangle \, d\sigma - \sum_{j=0}^{k-1} \int_{\mathfrak{M}} |\langle f_z, \psi^{(j)} \rangle|^2 \, d\sigma \geq 0$$

Then we have an upper bound on a sum involving eigenvalues.

Example: Recover Kröger's result

With the Parseval identity,

$$\int_{\mathfrak{M}} |\left\langle e^{i\mathbf{p}\cdot\mathbf{x}}, \psi^{(j)} \right\rangle|^2 = (2\pi)^d \|\psi^{(j)}\|^2 = (2\pi)^d.$$

IF $|\mathfrak{M}_0||\Omega| \geq (2\pi)^d k,$ then

$$(2\pi)^d \sum_{j=0}^{k-1} \mu_j \leq \int_{\mathfrak{M}_0} |\mathbf{p}|^2 |\Omega|$$

Choosing as a ball of radius R, a calculation gives Kröger.

Extensions to other homogeneous spaces

Recently, with El Soufi and Ilias we derived sharp inequalities for Neumann Laplacian eigenvalues on subdomains of homogeneous spaces, including extensions of Kröger for domains conformal to R^d, and analogues on compact such as

$$\sum_{j=0}^{k-1} \mu_j \leq \frac{|\Omega|}{|M|} \sum_{\ell=1}^{\left\lceil \sqrt{\frac{k|M|}{|\Omega|}} \right\rceil} \Lambda_\ell$$

with implications for Z(t). Cf. Strichartz, 1996.

Proof

Let $y_{l,1}, y_{l,2}, \cdots, y_{l,m_l}$ be an L^2-orthonormal basis of the eigenspace associated to Λ_l. It is well known that the function $\sum_{j=1}^{m_l} y_{l,j}^2$ is constant, that is

$$\sum_{j=1}^{m_l} y_{l,j}^2 = \frac{m_l}{|M|}. \tag{17}$$

Since $2|\nabla y_{l,j}|^2 = \Delta y_{l,j}^2 - \Lambda_l y_{l,j}^2$, it follows that the function $\sum_{j=1}^{m_l} |\nabla y_{l,j}|^2$ is also constant and that we have

$$\sum_{j=1}^{m_l} |\nabla y_{l,j}|^2 = \frac{m_l \Lambda_l}{|M|}. \tag{18}$$

Let ϕ be any $L^2(\Omega)$-normalized function. Then

$$\sum_{l=0}^{\infty} \sum_{j \le m_l} \langle y_{l,j}, \phi \rangle_{L^2(\Omega)}^2 = \|\phi\|_{L^2(\Omega)}^2 = 1.$$

Moreover, for any $L \in \mathbb{N}^*$,

$$\sum_{l \le L} \sum_{j \le m_l} \|y_{l,j}\|^2_{L^2(\Omega)} = \frac{|\Omega|}{|M|} \sum_{l \le L} m_l$$

and

$$\sum_{l \le L} \sum_{j \le m_l} \|\nabla y_{l,j}\|^2_{L^2(\Omega)} = \frac{|\Omega|}{|M|} \sum_{l \le L} m_l \Lambda_l$$

By the theorem,

$$\mu_k \left(\frac{|\Omega|}{|M|} D_L - k \right) \le \frac{|\Omega|}{|M|} \sum_{l \le L} m_l \Lambda_l + \frac{D_L}{|M|} \int_\Omega V dv_g - \sum_{j \le k} \mu_j(\Omega, V)$$

$$D_{L_k} = \sum_{l \le L_k} m_l \le \frac{|M|}{|\Omega|} k < \sum_{l \le L_k + 1} m_l = D_{L_k+1}.$$

Phase-space bounds

For a better result with a potential,

Use coherent states (like wavelets):

$$f_\zeta(\mathbf{x}) := \frac{1}{(2\pi)^{\nu/2}} e^{i\mathbf{p}\cdot(\mathbf{x})+\rho(\mathbf{x})} h(\mathbf{x}-\mathbf{y}).$$

resp.

$$f_\zeta(\mathbf{x}) := y_{\ell\,m}(\mathbf{x}) e^{\rho(\mathbf{x})} h_{\mathbf{y}}(\mathbf{x})$$

Cf. Lieb-Loss

Phase space estimates with V(x)

- The intuition in physics is that each eigenfunction fills a certain volume in phase space, $\{(\mathbf{x}, \mathbf{y}) \in \Omega \times R^{\nu}\}$

  Thus the number of energy levels (eigenvalues) $\leq \Lambda$ is proportional to the volume of the region in phase space for which

$$\{(\mathbf{p}, \mathbf{x}) : \ |\mathbf{p}|^2 + V(\mathbf{x}) \leq \Lambda\}$$

For domains conformal to Euclidean sets, we take

$$f_\varsigma(\mathbf{x}) := \frac{1}{(2\pi)^{\nu/2}} e^{i\mathbf{p}\cdot(\mathbf{x})+\rho(\mathbf{x})} h(\mathbf{x}-\mathbf{y}).$$

and reason as follows

Some definitions

- The *effective potential* $\widetilde{V}(\mathbf{x}) := V(\mathbf{x}) + |\nabla\rho|^2(\mathbf{x})$;
- The Euclidean *phase-space volume* for energy Λ,

$$\Phi(\Lambda) := |(\mathbf{x}, \mathbf{p}) : |\mathbf{p}|^2 + \widetilde{V}(\mathbf{x}) \le \Lambda| = \omega_\nu \int \left(\Lambda - \widetilde{V}(\mathbf{x})\right)_-^{\frac{\nu}{2}} d^\nu x,$$

- The *weighted phase-space volume* and *weighted total energy*

$$\Phi_w(\Lambda) = \omega_\nu \int \left(\Lambda - \widetilde{V}(\mathbf{x})\right)_-^{\frac{\nu}{2}} w(\mathbf{x}) d^\nu x$$

$$\begin{aligned} E_w(\Lambda) &:= \int_{\{(\mathbf{x},\mathbf{p}):|\mathbf{p}|^2+\widetilde{V}(\mathbf{x})\le\Lambda\}} \left(|\mathbf{p}|^2 + \widetilde{V}(\mathbf{x})\right) w(\mathbf{x}) d^\nu x \\ &= \frac{\nu}{\nu+2}\omega_\nu \int_\Omega \left(\Lambda - \widetilde{V}(\mathbf{x})\right)_-^{1+\frac{\nu}{2}} w(\mathbf{x}) d^\nu x. \end{aligned}$$

Some definitions

- The L^2-normalized ground-state Dirichlet eigenfunction for the ball of geodesic radius r in M will be denoted h_r and $\mathcal{K}(h_r) := \int_{B_r} |\nabla h_r(\mathbf{x})|^2 d^\nu x$. I.e., in this section where $M = \mathbb{R}^\nu$, h is a scaled Bessel function and

$$\mathcal{K}(h_r) = \frac{j^2_{\frac{\nu}{2}-1,1}}{r^2}.$$

- $L(\Lambda)$ will denote the maximal Lipschitz constant of $\widetilde{V}(\mathbf{x})$ on the region $\Omega \cap \{\mathbf{x} : \widetilde{V}(\mathbf{x}) \leq \Lambda\}$.

$$\langle \phi, f_\zeta \rangle = \mathfrak{F}[h(\mathbf{x} - \mathbf{y}) e^{-\rho(\mathbf{x})} \phi(\mathbf{x})],$$

where if Ω is a strict subset of $\mathbb{R}^\nu$, then ϕ is extended by 0 outside Ω. Thus, with the Parseval identity for the Fourier transform,

$$\begin{aligned} \int_{\mathbb{R}^{2\nu}} |\langle \phi, f_\zeta \rangle|^2 \, d^\nu p \, d^\nu y &= \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^\nu} h(\mathbf{x} - \mathbf{y})^2 \, |\phi|^2 \, e^{-2\rho} d^\nu y \, d^\nu x \\ &= \int_{\mathbb{R}^\nu} |\phi|^2 \, e^{-2\rho} d^\nu x \\ &= \|\phi\|^2. \end{aligned} \tag{9}$$

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The set $\mathfrak{M}_0$ in Theorem 1.1 must be taken large enough that

$$\begin{aligned} k &\le \int_{\mathfrak{M}_0} \|f_\zeta\|^2 d\sigma \\ &= \frac{1}{(2\pi)^\nu} \int_{\mathfrak{M}_0} h^2(\mathbf{x}-\mathbf{y}) e^{2\rho(\mathbf{x})-2\rho(\mathbf{x})} d^\nu x d^\nu p \, d^\nu y \\ &= \frac{1}{(2\pi)^\nu} |\mathfrak{M}_0|, \end{aligned}$$

in which case

$$\sum_{j=0}^{k-1} \mu_j \le \int_{\mathfrak{M}_0} \mathcal{E}(f_\zeta) d^\nu p \, d^\nu y$$

$$= \frac{1}{(2\pi)^\nu} \int_{\mathfrak{M}_0 \times \Omega} \left((|\mathbf{p}|^2 + V(\mathbf{x})) h^2(\mathbf{x}-\mathbf{y}) + |\nabla h(\mathbf{x}-\mathbf{y}) + h(\mathbf{x}-\mathbf{y}) \nabla \rho(\mathbf{x})|^2 \right) w(\mathbf{x}) d^\nu x \, d^\nu p \, d^\nu y$$

We now make the ansatz that $\mathfrak{M}_0 = \mathfrak{M}_0(\Lambda) := \{(\mathbf{x},\mathbf{p}) : \mathbf{x} \in \Omega, |\mathbf{p}|^2 + \tilde{V}(\mathbf{x}) \leq \Lambda\}$, where $\Lambda \geq \Lambda(k)$, defined as the minimum value of Λ for (10) to be valid.

Since the support of h is restricted to a ball of radius r, the x-integral may be restricted to the set $\{\mathbf{x} \in \Omega : \exists \mathbf{y}, |\mathbf{y}-\mathbf{x}| \leq r, |\mathbf{p}|^2 + \tilde{V}(\mathbf{x}) \leq \Lambda\} \subset \mathfrak{M}_0(\Lambda + L(\Lambda)r)$. Thus, integrating first in $\mathbf{y}$, the right side of (11) is bounded above by

$$\begin{aligned}
&\frac{1}{(2\pi)^\nu}\int_{\{\mathbf{p}:|\mathbf{p}|^2\leq\Lambda\}}\int_{\{\mathbf{x}:\tilde{V}(\mathbf{x})\leq\Lambda+L(\Lambda)r-|\mathbf{p}|^2\}}\int_{\mathbb{R}^\nu}\Big((|\mathbf{p}|^2+V(\mathbf{x}))h^2(\mathbf{x}-\mathbf{y})\\
&\qquad\qquad + |\nabla h(\mathbf{x}-\mathbf{y}) + h(\mathbf{x}-\mathbf{y})\nabla\rho(\mathbf{x})|^2\Big)\, w(\mathbf{x}) d^\nu y\, d^\nu x\, d^\nu p\\
&\quad = \frac{1}{(2\pi)^\nu}\int_{\{\mathbf{p}:|\mathbf{p}|^2\leq\Lambda\}}\int_{\{\mathbf{x}:\tilde{V}(\mathbf{x})\leq\Lambda+L(\Lambda)r-|\mathbf{p}|^2\}}\int_{\mathbb{R}^\nu}\Big((|\mathbf{p}|^2+\tilde{V}(\mathbf{x}))h^2(\mathbf{x}-\mathbf{y})\\
&\qquad\qquad + |\nabla h(\mathbf{x}-\mathbf{y})|^2 + \nabla\rho(\mathbf{x})\cdot\nabla h^2(\mathbf{x}-\mathbf{y})\Big)\, w(\mathbf{x}) d^\nu y\, d^\nu x\, d^\nu p.
\end{aligned}$$

The last contribution vanishes because

$$\int_{\mathbb{R}^\nu}\nabla\rho(\mathbf{x})\cdot\nabla h^2(\mathbf{x}-\mathbf{y}) d^\nu y = \int_{\mathbb{R}^\nu}\nabla\rho(\mathbf{x})\cdot\nabla 1 d^\nu y = 0, \tag{12}$$

leaving

$$\sum_{j=0}^{k-1}\mu_j \leq \frac{1}{(2\pi)^\nu}\int_{\mathfrak{M}_0(\Lambda+L(\Lambda)r)}(|\mathbf{p}|^2+\tilde{V}(\mathbf{x})+\mathcal{K}(h_r))w(\mathbf{x})d^\nu x\, d^\nu p \tag{13}$$

for all values of $r > 0$. This upper bound is of the form

$$\begin{aligned}
&\frac{1}{(2\pi)^\nu}\int_{\mathfrak{M}_0(\Lambda)}\Big(|\mathbf{p}|^2+\tilde{V}(\mathbf{x})\Big)\, w(\mathbf{x})d^\nu x\, d^\nu p\\
&\qquad + |\mathfrak{M}_0(\Lambda+L(\Lambda)r)|\frac{j^2_{\frac{\nu}{2}-1,1}}{r^2} + \frac{1}{(2\pi)^\nu}\int_{\mathfrak{M}_0(\Lambda+L(\Lambda)r)\setminus\mathfrak{M}_0(\Lambda)}(|\mathbf{p}|^2+\tilde{V}(\mathbf{x}))w(\mathbf{x})d^\nu x\, d^\nu p
\end{aligned}$$

The ultimate bound is of the form of the "expected" phase-space quantity, plus explicit corrections that are of lower order in k.

Extensions to traces of concave functions of λ_j and to partition functions

Here we have three tricks:

1. Laplace transform
2. Karamata's inequality on dominating sequences
3. Jensen's inequality.


THE END