The lightest coins and the lightest rolling bearings

Colloquium
23 March 2004
Kuwait University

Abstract: To be used in a vending machine, a coin must fully fit into a slot no matter how it is turned, but it need not be circular. In the early 20th century Lebesgue and Blaschke figured out the planar convex domain of given constant width and the smallest possible volume. That is, they found the lightest possible silver coin (no holes, fixed thickness) that can be used in a vending machine. For nearly a century nobody has answered the same question in three dimensions - what is the lightest possible rolling bearing? I'll discuss some approaches to this problem and both old and new conjectures

A direct proof of a theorem of Blaschke and Lebesgue, Journal of Geometric Analysis 12(2002)81-88.

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FIFTY PENCE
50
TWENTY PENCE
1995
20
WS
PU
X
50
PENCE
GIVE
WOMEN
THE
VOTE
2003
1903


1
DALASI


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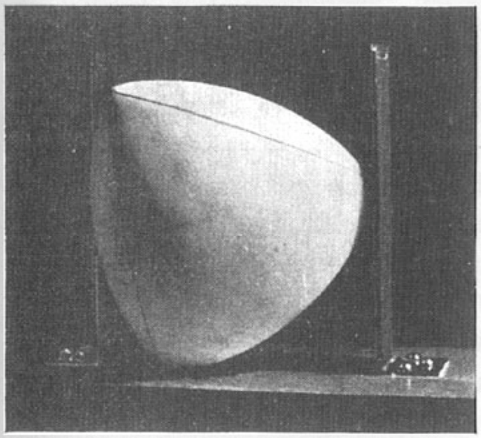
CANADA
NICKEL
1751-1951
5
CENTS


20
1998
CENTS
10
CENTS 1998
5
CENTS 1998

(From Hilbert & Cohn Vossen)



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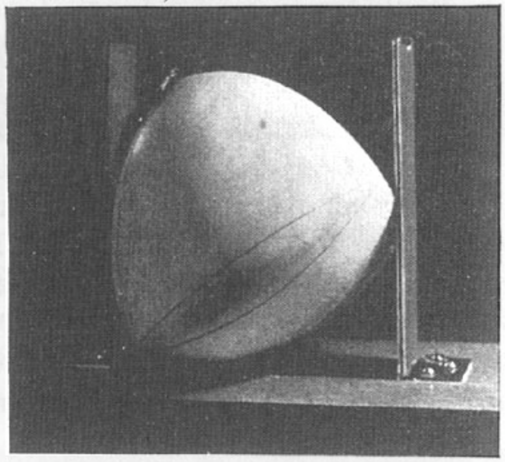

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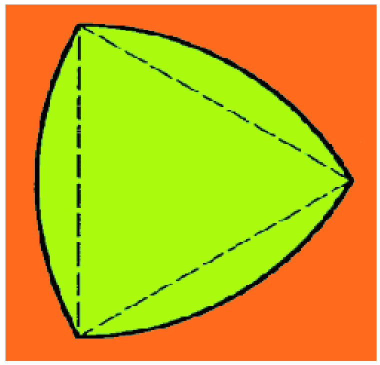
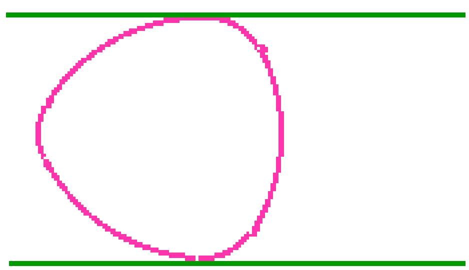
Some history

Euler, 1778, Acta of the Petersburg Academy,
first appearance of the Reuleaux triangle.



Wolf steam engine, 1830, piston in shape of Reuleaux triangle

Reuleaux, 1876, Kinematics of Machinery, various rollers



Minkowski, 1904, First systematic mathematical treatment of bodies of constant width.

Meissner, 1912, plaster model of 3-D bodies of constant width

Lebesque, 1914, 1921, Reuleaux triangle is smallest

Blaschke, 1915, rigorous proof

(Other proofs by Fujiwara, 1931; Eggleston, 1952; Besicovich, 1963)

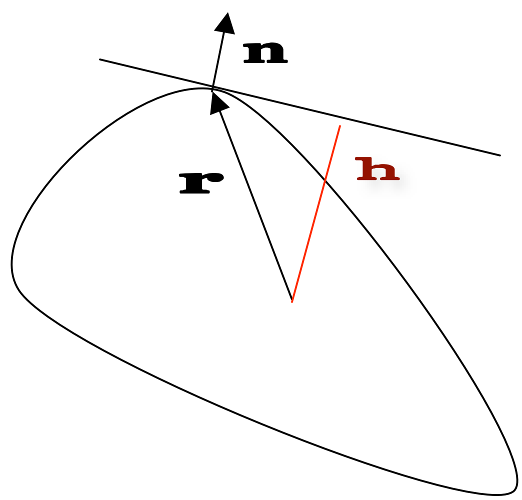
How best to describe the surface of a convex body D?

r(s)   - an embedded curve.

support function h(θ) for D - ∂D is a continuous image of S^{d-1} .

h(θ) = **r•n**

n =



Example: The Reuleaux triangle

How are **r** and h connected?

$$h(\theta) = \mathbf{r} \cdot \mathbf{n} = \mathbf{r}^T R_{\pi/2} \dot{\mathbf{r}}$$

Conversely, since

$$\frac{dh}{d\theta} = R\,(\mathbf{t} \cdot \mathbf{n} + \mathbf{r} \cdot \kappa\, \mathbf{t}) = \mathbf{r} \cdot \mathbf{t},$$

$$\mathbf{r} = (\mathbf{r} \cdot \mathbf{t})\,\mathbf{t} + (\mathbf{r} \cdot \mathbf{n})\,\mathbf{n} = h'(\theta)\,\mathbf{t} + h(\theta)\,\mathbf{n}$$

Need to describe:

The shape - either **r** or h will suffice

The constraint (constant width)

The objective (volume/area)

ω^a := antipode to ω. (I.e., $\theta \to \theta + \pi$)

$B = h(\omega^a) + h(\omega)$

$$\mathrm{Vol}(D) = \frac{1}{d}\int h(\omega)\,dS$$

$$= \frac{1}{d}\int h(\omega)\frac{d\omega}{\prod \kappa_j}$$

Radii of curvature

$$R_j = 1/\kappa_j$$

The objective functional:

$$\mathrm{Vol}(D) = \frac{1}{d}\left\langle h, \prod R_j \right\rangle_{S^{d-1}}$$

What is the relationship between h and R_j?

$$\nabla^2_{S^{d-1}} h + (d-1)\, h = \sum_{j=1}^{d-1} R_j =: R$$

Implication of the constraint $h + h^a = B$

$$R + R^a = (d\text{-}1)\, B$$

Can R be used as the control variable? The number d-1 in

$$\nabla^2_{S^{d-1}} h + (d-1)\, h = R$$

Is an eigenvalue of the Laplacian on a sphere.

E.g., in d=2, the Laplacian is $d^2/d\theta^2$, but

$$(d^2/d\theta^2 + 1) \sin(\theta) = 0.$$

“Second Fredholm alternative”:

There is a solution if and only if R is orthogonal to the eigenspace, and it is unique if we insist that h is also orthogonal to the eigenspace

3 Dimensions. From the curvature to the support function and the body

When *is* there a convex body?

Let's first confront the issue of when there is a convex body corresponding to a possible curvature function. In two dimensions the condition that a curve be closed is rather easily stated in terms of the curvature function, but the issue of what curvature functions correspond to properly embedded convex bodies in higher dimensions was long mysterious. The problem was analyzed in by W.J. Firey [The determination of convex bodies from their mean radius of curvature functions, Mathematika **14**(1967)1-13; Christoffel's problem for general convex bodies, *Ibid.* **15**(1968)7-21], who found necessary and sufficient conditions. The key condition is that for any pair of unit vectors U and V,

(||U||+||V||-||U+V||) Integrate[R] >= Integrate[R Lambda],

where the kernel function Lambda is defined as follows.

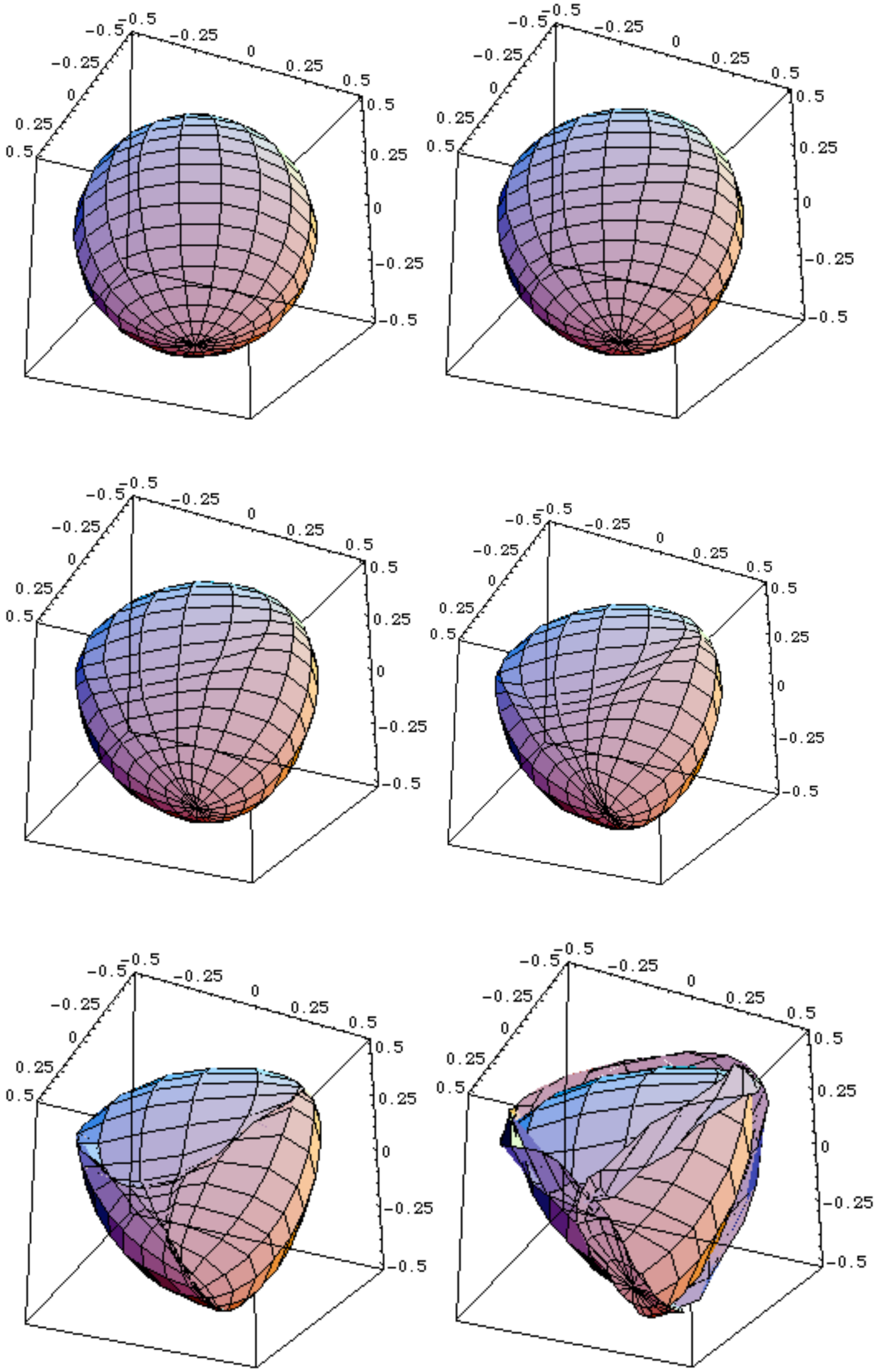
```
LambdaStar[U_, V_, theta_, phi_] :=
 U.{Sin[theta] Cos[phi], Sin[theta] Sin[phi], Cos[theta]}
  Ln[
   (1 - (U + V).{Sin[theta] Cos[phi], Sin[theta] Sin[phi], Cos[theta]}/
       Sqrt[(U + V).(U + V)]) /
    (1 - U.{Sin[theta] Cos[phi], Sin[theta] Sin[phi], Cos[theta]}/
       Sqrt[U.U])]
```

```
Lambda[U_, V_, theta_, phi_] :=
 LambdaStar[U, V, theta, phi] + LambdaStar[V, U, theta, phi]
```

```
Integrate[Lambda[{0, 0, 1}, {0, 1, 0}, t, p], {t, 0, Pi/2}]
```

$$\int_0^{\frac{\pi}{2}} \left(\mathrm{Cos}[t]\ \mathrm{Ln}\left[\frac{1-\frac{\mathrm{Cos}[t]+\mathrm{Sin}[p]\ \mathrm{Sin}[t]}{\sqrt{2}}}{1-\mathrm{Cos}[t]}\right] + \mathrm{Ln}\left[\frac{1-\frac{\mathrm{Cos}[t]+\mathrm{Sin}[p]\ \mathrm{Sin}[t]}{\sqrt{2}}}{1-\mathrm{Sin}[p]\ \mathrm{Sin}[t]}\right] \mathrm{Sin}[p]\ \mathrm{Sin}[t] \right) dt$$

Unfortunately, the kernel function Lambda proves to be rather difficult to work with, either analytically or numerically.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