Mathematics of Quantum Mechanics on Thin Structures

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'You may seek it with trial functions---and seek it with care;
    You may hunt it with rearrangements and hope;
    You may perturb the boundary with a lump here and there;
    You may fool it with some series rope-a-dope---'

Apologies to Lewis Carroll.
The postulates of quantum theory

Every “observable” is modeled by a self-adjoint operator $\langle A \phi, \psi \rangle = \langle \phi, A\psi \rangle$ on a Hilbert space. (Complete, normed, linear space. Usually $L^2(\Omega)$, $\langle f, g \rangle := \int_{\Omega} f(x)\overline{g(x)}dV$. )
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- The possible measurements are $\text{sp}(A)$
  - If $A$ has discrete eigenvalues, it is “quantized.”
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The state of the system is defined by a vector that has been normalized:

\[
\|\psi\|^2 = \langle \psi, \psi \rangle = 1
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  \[
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  \]

- Expectation values: \( E(f(A)) = \langle (A) \psi, \psi \rangle \)
How do things change in time?

There is a Hamiltonian operator, corresponding to the total energy: $H$, which is a function of momentum $p$ and position $x$.

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi.$$  
$$\psi(t) = e^{-iHt} \psi(0)$$  
$$H \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi(x) + V(x)\psi(x)$$
A brief course on spectral theory

- A self-adjoint or Hermitian operator on a Hilbert space is a linear mapping with the property that

\[ \langle A\phi, \psi \rangle = \langle \phi, A\psi \rangle \]

- This means \( A = A^* \), the adjoint operator

- \( D(A) \) is a dense set but not all of \( H \) if \( A \) is unbounded. For a differential operator, different boundary conditions (Dirichlet, Neumann) correspond to different choices of \( D(A) \).
A brief course on spectral theory

- Hermitian matrices are the self-adjoint operators on the finite-dimensional Hilbert space $\mathbb{C}^n$. The algebraic and analytical properties of self-adjoint operators are similar to those of Hermitian matrices.
  - Diagonalization, orthogonal basis
  - Noncommutative algebra, commutative subalgebra
A brief course on spectral theory

- The *resolvent set* $\rho(H)$ is the set of complex numbers $z$ such that $(H - z)$ has an inverse, known as the *resolvent operator*, $(H - z)^{-1}$, which is continuous.

- The spectrum $\text{sp}(H)$ is the complement of $\rho(H)$. The spectrum is a closed set, and if $H$ is self-adjoint, $\text{sp}(H)$ is real.
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- The spectrum $sp(H)$ is the complement of $\rho(H)$. The spectrum is a closed set, and if $H$ is self-adjoint, $sp(H)$ is real.

- 2 possibilities: 1. $(H - z)^{-1}$ doesn’t exist; or 2. it exists, but not bounded.
A brief course on spectral theory

- An eigenvalue $\lambda$ of $H$ belongs to $\text{sp}(H)$, because if $(H - \lambda) u = 0$ for some $u \neq 0$, then $(H - \lambda)$ is not injective. $(H - z)^{-1}$ doesn’t exist.

- Eigenvalues are not the only numbers in the spectrum. If a point of the spectrum of a s-a op. is isolated, however, it is an eigenvalue.
von Neumann’s Spectral Theorem

Suppose $A = A^*$ and $\phi \in \mathcal{H}$, $\phi \in D(A^n)$, $n = 0 \cdots N$. There exists a measure $d\mu_\phi$ supported on $\text{sp}(A)$ such that $\langle \phi, A^n \phi \rangle = \int_{\mathbb{R}} \lambda^n d\mu_\phi(\lambda)$. (In particular, $\|\phi\|^2 = \int_{\mathbb{R}} d\mu_\phi$.)

Moreover, for any bounded function $f$ that is continuous on $\text{sp}(A)$, there is a unique bounded operator $f(A)$ such that for any such $\phi$, $\langle \phi, f(A) \phi \rangle = \int_{\mathbb{R}} f(\lambda) d\mu_\phi(\lambda)$. The algebra $\{f(A)\}$ is isomorphic to $C[\text{sp}(A)]$, and

$$\|f(A)\|_{\text{sp}} = \sup_{\text{sp}(A)} |f(\lambda)|.$$
Corollary: Rayleigh-Ritz

\[ \inf sp(A) = \inf_{\|\varphi\|=1} \langle \varphi, A\varphi \rangle = \inf \int \lambda d\mu_\varphi \]

(Infimum over probability measures supported on \(sp(A)\).)
Corollary: Rayleigh-Ritz

\[ \inf sp(A) = \inf_{\|\varphi\|=1} \langle \varphi, A\varphi \rangle = \inf_{d\mu_\phi} \int \lambda d\mu_\phi \]

For \( H = -\nabla^2 + V(x) \), \( \varphi \) smooth,

\[ \inf sp(H) \leq \frac{\int |\nabla \varphi|^2 + V(x)|\varphi|^2}{\int |\varphi|^2} \]

"Rayleigh quotient."
Three perspicuous examples of spectral analysis

1. Let $A$ be an $n\times n$ matrix and $\Phi$ an $n$-vector. Choose a basis of normalized eigenvectors $\{e_j\}$, so $A e_j = \lambda_j e_j$.

   - Given any function $f$ defined on $\{\lambda_j\}$, $f(A)$ is the diagonal matrix defined by
     
     $f(A) e_j = f(\lambda_j) e_j$.

   - The measure $\mu_\Phi$ associated with
     
     $\Phi = \sum \Phi_j e_j$ is
     
     $\sum |\Phi_j|^2 \delta_{\lambda_j}$
     
     For instance, $\Phi \cdot A \Phi = \sum |\Phi_j|^2 \lambda_j$
Three perspicuous examples of spectral analysis

1. Rayleigh-Ritz estimates the lowest eigenvalue from above:

\[ \Phi \cdot A \Phi = \sum |\Phi_j|^2 \lambda_j \geq \lambda_1 \sum |\Phi_j|^2 \]
Three perspicuous examples of spectral analysis

2. Let $-\Delta$ be the Laplace operator on the set $L^2(\mathbb{R}^d)$ of square-integrable functions. There are no eigenfunctions that can be normalized in $L^2$, but we can “diagonalize” $-\Delta$ with the Fourier transform.

$$\mathcal{F} [f] (\mathbf{k}) = \hat{f}(\mathbf{k}) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\mathbf{k} \cdot \mathbf{x}} f(\mathbf{x}) d\mathbf{x}$$

$$\mathcal{F}^{-1} [g] (\mathbf{x}) = \check{g}(\mathbf{x}) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{+i\mathbf{k} \cdot \mathbf{x}} g(\mathbf{k}) d\mathbf{k}$$
Three perspicuous examples of spectral analysis

2. Because for any “test function” $\phi$,

$$\mathcal{F} \left[ \frac{\partial \phi}{\partial x_\alpha} \right] (k) = k_\alpha \hat{\phi}(k),$$

$$\mathcal{F} [-\Delta \phi] k = |k|^2 \hat{\phi}(k),$$

and hence for any function $f$

$$\mathcal{F} [f(-\Delta)\phi] k = f(|k|^2) \hat{\phi}(k).$$
Three perspicuous examples of spectral analysis

2. Rayleigh-Ritz tells us the spectrum, while continuous, is nonnegative:

\[
\inf \text{sp}(-\Delta) = \inf_{\|\varphi\|=1} \langle \hat{\varphi}, |k|^2 \hat{\varphi} \rangle \geq 0
\]
Three perspicuous examples of spectral analysis

2. If \( u_t = \Delta u \), and we think of \( u \) as a finite-dimensional vector and \(-\Delta\) as a number \( \geq 0 \), then

\[
u(t) = \exp(-(-\Delta)t) \, u(0).
\]

With the spectral theorem we define \( \exp(-(-\Delta)t) \) as a bounded operator for any \( t \), and recover Fourier’s solution:

\[
u(t, x) = \mathcal{F}^{-1} \left[ e^{-|k|^2 t} \hat{u}(t, k) \right].
\]
Three perspicuous examples of spectral analysis

2. Exercise: Given a function $\Phi$, what is the associated measure on $\text{sp}(-\Delta) = \{\lambda: \lambda \geq 0\}$?

**Hint:** Work out the form of $\mu_\Phi$ by equating $\lambda$ with $|k|^2$ and changing variables.
Three perspicuous examples of spectral analysis

3. Find the eigenvalues and eigenvectors of the Laplacian on a 2D ring, $1 \leq r \leq 1+\delta$.

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

Because of the symmetry, variables separate, and the eigenfunctions are products of $\sin(m \theta)$ with eigenfunctions of an ODE in $r$, which is a version of Bessel’s eqn
Three perspicuous examples of spectral analysis

3. From Kuttler and Sigillito, 1984:

The annulus $a \leq r \leq b$ has eigenfunctions

\[
(3.8) \quad u_{m,n} = \left[ Y_m (k_{m,n}) J_m \left( \frac{k_{m,n} r}{a} \right) - J_m (k_{m,n}) Y_m \left( \frac{k_{m,n} r}{a} \right) \right] [A \cos m\theta + B \sin m\theta],
\]

where $Y_m$ is the $m$th Bessel function of the second kind. The eigenvalues are

\[
(3.9) \quad \lambda_{m,n} = \left( \frac{k_{m,n}}{a} \right)^2, \quad m = 0, 1, 2, \ldots, \quad n = 1, 2, \ldots,
\]

where $k_{m,n}$ is the $n$th root of

\[
(3.10) \quad Y_m (k) J_m \left( \frac{kb}{a} \right) - J_m (k) Y_m \left( \frac{kb}{a} \right) = 0.
\]

Tables of these roots are given in [66]. Sectors of an annulus can also be treated similarly.
### Table 34. The first six Roots

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<th>k</th>
<th>(k - 1) x₀,1</th>
<th>(k - 1) x₁,1</th>
<th>(k - 1) x₂,1</th>
<th>(k - 1) x₃,1</th>
<th>(k - 1) x₄,1</th>
<th>(k - 1) x₅,1</th>
<th>(k - 1) x₆,1</th>
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</tbody>
</table>

Jahnke-Emde-Lösch 1960
ParametricPlot3D[{r Cos[t], r Sin[t], BesselJ[4, 11.065 r] Cos[4 t]}, {r, 1, 1.592}, {t, 0, 2 Pi}]
What are the eigenvalues when the ring is thin, i.e., $a = 1$, $b = 1+\delta$?
Zinc oxide quantum wire loop

Z.L. Wang, Georgia Tech
What are the eigenvalues when the ring is thin, i.e., $a = 1$, $b = 1 + \delta$?

**ANS:** $m^2 \pi^2 / \delta^2 + 4 n^2 \pi^2 - 1/4 + O(\delta)$

$n = 0, \pm 1, \pm 2, \ldots; \ m = 1, 2, \ldots$

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$
What are the eigenvalues when the ring is thin, i.e., \( a = 1, b = 1+\delta \)?

**ANS:** \( m^2 \pi^2 / \delta^2 + 4 n^2 \pi^2 -1/4 + O(\delta), \)

\( n = 0, \pm 1, \pm 2, \ldots; m = 1, 2, \ldots \)

Eigenvalues of the interval (0,\( \delta \)) (Dirichlet BC)

Eigenvalues of the ring (periodic BC)

To \( O(\delta) \) it is as if the variables \( \theta \) and \( r \) separate as *Cartesian* variables. *Except that there is a correction -1/4. This is the effect of curvature.*
Thin structures and local geometry

Thin domain of fixed width
variable \( r \) = distance from edge

Energy form in separated variables:

\[
\int_D \left| \nabla_{\parallel} \xi \right|^2 d^{d+1}x + \int_D |\xi_r|^2 d^{d+1}x
\]
The result:

\[-\nabla^2_{||} + q(x) = -\Delta_\Omega + q(x)\]

\[q(x) = \frac{1}{4} \left( \sum_{j=1}^{d} \kappa_j \right)^2 - \frac{1}{2} \sum_{j=1}^{d} \kappa_j^2\]

which reduces to \(-1/4\) when \(\Omega\) is a circle of radius 1.
Weapons used when hunting for eigenvalues

- Rayleigh-Ritz
- Other variational principles, such as min-max
- Approximate eigenvalues
- Perturbation theory
  - If an eigenvalue and eigenvector are known for $A$, solve
    \[ (A + \kappa B)u(\kappa) = \lambda(\kappa)u(\kappa) \]
    with power series in $\kappa$. 
Fun with Rayleigh and Ritz
How does a spectral theorist estimate $\pi^2$?

We know $-\frac{d^2}{dx^2} \sin(n\pi x) = n^2 \pi^2 \sin(n\pi x)$, vanishing BC at 0, 1, $\lambda_n = n^2 \pi^2$.

According to Rayleigh-Ritz,

$$\pi^2 \leq \frac{\langle \varphi, \frac{-d^2}{dx^2} \varphi \rangle}{\langle \varphi, \varphi \rangle} = \frac{\int_0^1 |\varphi'(x)|^2 dx}{\int_0^1 |\varphi(x)|^2 dx}.$$ 

The sine function is transcendental. It would be nicer if $\varphi$ were a polynomial.

Let’s try $\varphi(x) = x(1-x)$, $0 \leq x \leq 1$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{graph.png}
\end{figure}
How does a spectral theorist estimate $\pi^2$?

$$\pi^2 \leq \frac{\int_0^1 (1 + 4x^2 - 4x) \, dx}{\int_0^1 (x^2 + x^4 - 2x^3) \, dx} = \frac{1 + 4/3 - 2}{1/3 + 1/5 - 1/2} = 10$$

Cf. 9.8696044. 10 is too large by 1.32%.
What about a lower bound?

Rayleigh and Ritz like to guess that an eigenvalue is

$$\lambda_k \approx \eta_k := \langle \varphi_k, H \varphi_k \rangle,$$

where the trial function is chosen cleverly. For \( \lambda_1 \) this is always a bit too high. If the eigenvalue is known to be isolated, Temple’s inequality offers a counterpart:

$$\lambda_1 \geq \eta_1 - (\langle H \varphi_k, H \varphi_k \rangle - \eta_1^2)/(\text{isolation from above})$$
Kato’s Improvement of Temple’s Inequality

$A$ is a self adjoint operator such that $\text{sp}(A) \cap (\alpha, \beta) = \lambda_*$. For $\varphi \in D(A)$, $\|\varphi\| = 1$, $|\eta| = \langle \varphi, A\varphi \rangle$, $\epsilon^2 = \|A\varphi\|^2 - \eta^2$. Then

$$\epsilon^2 \leq (\beta - \eta)(\eta - \alpha) \implies \eta - \frac{\epsilon^2}{\beta - \eta} \leq \lambda_* \leq \eta + \frac{\epsilon^2}{\eta - \alpha}$$

**Proof.** EXERCISE: Hint:

1. Assume $\text{sp}(A) \cap (\alpha', \beta') = \phi$. Then $\nu \in \text{sp}(A) \Rightarrow \left| \nu - \frac{\alpha + \beta'}{2} \right| \geq \frac{\beta' - \alpha'}{2}$

2. Square and show $\nu^2 - \eta^2 \geq (\alpha + \beta')\nu - \alpha\beta' - \eta^2$. If $\|\varphi\| = 1$, $d\mu_\phi$ is a probability measure, and by integrating

$$\epsilon^2 \geq (\alpha' + \beta')\eta - \alpha'\beta' - \eta^2 = (\eta - \alpha)(\beta - \eta)$$

Contradicting assumption.

3. Given, $\epsilon$, $\eta$, $\alpha' = \alpha$, choose $\beta'$ optimally. Given $\epsilon$, $\eta$, $\beta' = \beta$, choose $\alpha'$ optimally.
How does a spectral theorist estimate $\pi^2$?

What About a Lower Bound?

Temple’s inequality. We know $\pi > 3$, so $\lambda_2 \geq 4 \cdot 9 = 36$.

$$
\lambda_1 \geq \langle \varphi, A\varphi \rangle - \frac{\|A\varphi\|^2 - \langle \varphi, A\varphi \rangle^2}{36 - \langle \varphi, A\varphi \rangle} = 10 - \frac{1}{\phi} \left( \frac{\int_0^1 (\varphi'')^2 \, dx}{26} \right) - 100
$$

$$
= 10 - \frac{120 - 100}{26} \simeq 9.23.
$$

Improvement. Project into subspace of functions even under $x \leftrightarrow 1 - x$. The second eigenvalue becomes $\lambda_3 > 9 \cdot 9$, and

$$
\lambda_1 \geq 10 - \frac{120 - 100}{71} \simeq 9.72. \quad \text{(Too small by 1.5%)}
$$
Weapons used when hunting for eigenvalues

- Rayleigh-Ritz
- Other variational principles, such as min-max

\[ \lambda_k = \max \min \langle \varphi, A \varphi \rangle, \ \varphi \text{ normalized and orthogonal to a } k-1 \text{ dimensional subspace.} \]
Weapons used when hunting for eigenvalues

- Rayleigh-Ritz
- Other variational principles, such as min-max
- Approximate eigenvalues, i.e., sequences

\[ \|\varphi_n\| = 1, \text{ such that } \|(H - \lambda)\varphi_n\| \to 0. \]
Weapons used when hunting for eigenvalues

- Rayleigh-Ritz
- Other variational principles, such as min-max
- Approximate eigenvalues
- Perturbation theory
  - If an eigenvalue and eigenvector are known for $A$, solve
    \[(A + \kappa B)u(\kappa) = \lambda(\kappa)u(\kappa)\]
    with power series in $\kappa$. 
1. If the volume is fixed, what shape minimizes the fundamental eigenvalue?

Answer: The ball. Conjectured by Rayleigh, proved by Faber and Krahn.

A modern proof combines the Rayleigh-Ritz inequality, the notion of rearrangement, the “co-area formula,” and the isoperimetric inequality.
2 Cases where $\Omega$ is annular and $\lambda_1$ is maximized by the circularly symmetric case.

\[ g \rightarrow \|dx\|^2 = dr^2 + (1 + kr)^2 dt^2 \]

$\kappa(t)$ - curvature m edge.

\[ \int \nabla \varphi^2 \, d\chi = \int_0^S \int_0^1 \left( \varphi_r^2 + \frac{\varphi_t^2}{(1 + kr)^2} \right) (1 + kr) \, dt \, dr \]

Rayleigh quotient with $\varphi$ independent of $t$.

$\lambda_1 \leq \int_0^S \int_0^1 \varphi_r^2 (1 + kr) \, dt \, dr = \int_0^S \varphi_r^2 (1 + 2\pi r) \, dr = \lambda_1^* \text{ if } \varphi \rightarrow U_1(\text{ann})$
Spherical Shells. Infinitesimal case.

Suppose $\Omega \equiv S^2 \subset \mathbb{R}^3$, $\lambda_1, \ldots, \lambda_n$ $(-\Delta + q(K))$.

$q(K) = -\frac{(K_1 - K_2)^2}{4}$ \quad (similar for $-\frac{K_1 K_2}{4}$, etc.)

$\lambda_1 \leq \frac{\|\nabla 1\|^2 + \int q(K)}{\int 1} = 0 + \langle q \rangle \leq 0$

= iff sphere. (WHY?)
Same argument for $\Omega = \text{circle}$

- $q(\kappa) = -\kappa^2/4$
- $\lambda_1 \leq -\langle \kappa^2 \rangle/4 \leq -\langle \kappa \cdot 1 \rangle^2/4$ (Cauchy-Schwarz)
  
  \[ = -\pi^2 \]

- Equality iff sphere. (Why?)
Proof outline for the Faber-Krahn theorem
Spherical rearrangement

Any (say, continuous, non-negative, compactly supported) function \( f(x) \) can be rearranged to a radially decreasing \( f^*(x) \) so that

\[
\mu(f^*(x) \geq h) = \mu(f(x) \geq h).
\]

Integrals of \( g(f(x)) \) are unaffected by this rearrangement.
Co-area formula

- Given a (say, smooth, non-negative, compactly supported) function \( f(x) \) on \( \Omega \) (and assuming \( \nabla f = 0 \) only on a null subset), what happens when you change variables to include \( h = f(x) \)?

- Ans: \( dV = \frac{dS(f^{-1}(h))}{|\nabla f|} \, dh \)

- Area on “level set”
Faber-Krahn

What happens to a Dirichlet integral when you rearrange the function?

Consider the quantity

$$I = \int_{\Omega} \int_{f^{-1}(h)} \psi \, ds \, dh$$
Faber-Krahn

\[ I = \int_{0}^{1} \int_{f^{-1}(h)}^{1} ds \, dh \]

The Cauchy-Schwarz-Bunyakovsky inequality states

\[ (\int g \, h \, du)^2 \leq \int g^2 du \cdot \int h^2 du, \]

so

\[ I^2 \leq \int_{0}^{1} \int_{f^{-1}(h)}^{1} |\nabla f|^2 ds \, dh \cdot \int_{f^{-1}(h)}^{1} \frac{1}{|\nabla f|^2} ds \, dh \]

\[ = \int_{\Omega} |\nabla f| ^2 \, dV \cdot \text{Vol} (\mathbb{S}) \]
Meanwhile, since \( f^{-1}(h) \) is the surface bounding the set \( \{ x : f(x) \geq h \} \), the volume of which is unchanged by symmetrization,

\[
\int_{f^{-1}(h)} ds \geq \int_{f^{*1}(h)} ds \quad \text{(isoperimetric thm)}.
\]

\[
\therefore I^2 \geq I^{*2} = \int_{\mathbb{R}^2} 17 + f^2 \, dv \cdot \text{Vol}(\mathbb{R})
\]
Yet another “isoperimetric theorem,” this time for $\lambda_2$.

Consider the thin-domain operator on a closed, simply connected surface in $\mathbb{R}^3$,

$$\nabla^2 - (\kappa_2 - \kappa_1)^2/4.$$

The ground state is maximized (at 0) by the sphere. Let’s fix the area and ask after the maximum of the second eigenvalue.
Yet another “isoperimetric theorem,” this time for $\lambda_2$.

Eigenfunctions of a self-adjoint operator, with different eigenvalues, are orthogonal. Therefore if we search over $\varphi$ orthogonally to $u_1$,

$$\lambda_2 \leq \langle \varphi, A \varphi \rangle / \| \varphi \|^2.$$
Yet another “isoperimetric theorem,” this time for $\lambda_2$.

Eigenfunctions of a self-adjoint operator, with different eigenvalues, are orthogonal. Therefore if we search over $\varphi$ orthogonally to $u_1$,

$$\lambda_2 \leq \langle \varphi, A \varphi \rangle / \|\varphi\|^2.$$ 

Problem: We don’t know $u_1$ a priori. One way around this is a lemma of J. Hersch:
Yet another “isoperimetric theorem,” this time for $\lambda_2$.

Lemma. (J. Hersch). Let $\Omega$ be a two-dimensional, closed, smooth Riemannian manifold of the topological type of the sphere, and specify a bounded, positive, measurable function $\rho$ on $\Omega$. Then there exists a conformal transformation $\Phi : \Omega \to S^2 \subset \mathbb{R}^3$, embedded in the standard way as the unit sphere, such that

$$
(3) \quad \int_{S^2} x\rho(\Phi^{-1}(x))Jd\hat{S} = 0.
$$
Yet another “isoperimetric theorem,” this time for $\lambda_2$.

For the trial function $\varphi$ let’s choose one of the Cartesian coordinates $x,y,z$ on $S^2$, but “pull back” to $\Omega$ with the inverse of Hersch’s conformal transformation. Let the resulting functions on $\Omega$ be called $X,Y,Z$. What do we know about $X,Y,Z$?
Yet another “isoperimetric theorem,” this time for $\lambda_2$.

For the trial function $\varphi$ let’s choose one of the Cartesian coordinates $x, y, z$ on $S^2$, but “pull back” to $\Omega$ with the inverse of Hersch’s conformal transformation. Let the resulting functions on $\Omega$ be called $X, Y, Z$. What do we know about $X, Y, Z$?

1. The functions $X, Y, Z$ are orthogonal, because the functions $x, y, z$ are orthogonal on $S^2$.
   
   * Note: The restrictions of $x, y, z$ to $S^2$ are the spherical harmonics = eigenfunctions:
   - $\nabla^2 x = 2 \ x$,
   - $\nabla^2 y = 2 \ y$,
   - $\nabla^2 z = 2 \ z$, 
Yet another “isoperimetric theorem,” this time for $\lambda_2$.

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1. The functions $X,Y,Z$ are orthogonal.

2. $X^2 + Y^2 + Z^2 = 1$, because $x_1^2 + x_2^2 + x_3^2 = 1$. 
Yet another “isoperimetric theorem,” this time for $\lambda_2$.

For the trial function $\varphi$ let’s choose one of the Cartesian coordinates $x,y,z$ on $S^2$, but “pull back” to $\Omega$ with the inverse of Hersch’s conformal transformation. Let the resulting functions on $\Omega$ be called $X,Y,Z$. What do we know about $X,Y,Z$?

1. The functions $X,Y,Z$ are orthogonal.
2. $X^2 + Y^2 + Z^2 = 1$, because $x_1^2 + x_2^2 + x_3^2 = 1$.
3. Identifying now $\rho$ with $u_1$, 
   \[<X,u_1> = \int_{S^2} x\rho(\Phi^{-1}(x))Jd\hat{S} = 0.\] Likewise for $Y, Z$. 
Ready to roll with Rayleigh and Ritz:

Let’s choose the trial function in

$$R(\zeta) := \frac{\int_{\Omega} |\nabla \zeta|^2 dS - \frac{1}{4} \int_{\Omega} (\kappa_2 - \kappa_1)^2 |\zeta|^2 dS}{\int_{\Omega} |\zeta|^2 dS}$$

as $\zeta = X, Y, \text{or} Z$. Considering for example $X$, conformality implies that

$$\int_{\Omega} |\nabla X|^2 dS = \int_{S^2} |\nabla x|^2 d\hat{S} = \frac{8\pi}{3}$$
Ready to roll with Rayleigh and Ritz:

Observing that

\[ a \leq \frac{b_j}{c_j} \]

\[ \Rightarrow \]

\[ a \leq \frac{\sum_j b_j}{\sum_j c_j} : \]

\[ \lambda_2 \leq \frac{8\pi - \int_\Omega (\kappa_2 - \kappa_1)^2 dS}{\int_\Omega 1 dS}. \]

Equality iff sphere. Why?