# Agmon metrics, exponential localization and the shape of quantum graphs

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Montréal
Dec.,2015

# *Abstract*

We show that the Agmon method for establishing exponential decrease of eigensolutions (or subsolutions) can be adapted to quantum graphs. As a generic matter, the rate of decay is controlled by an Agmon metric related to the classical Liouville-Geen estimate for the line, but more rapid decay is typical, arising from the geometry of the graph. We provide additional theorems capturing this effect with alternative Agmon metrics, one adapted to a path and the other using averaging.

This is joint work with Anna Maltsev of the University of Bristol, http://arxiv.org/abs/1508.06922

# Why do eigenfunctions localize?

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1. The tunneling effect.
2. Randomness (Anderson localization)

# *Quantum graphs*

- Microelectronic circuits modeled with ODES on *metric* graphs.

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- Applications in many other fields, including neuroscience and botany.
- Some 1D and some multi-D aspects.
- “Can one hear the graph structure?”

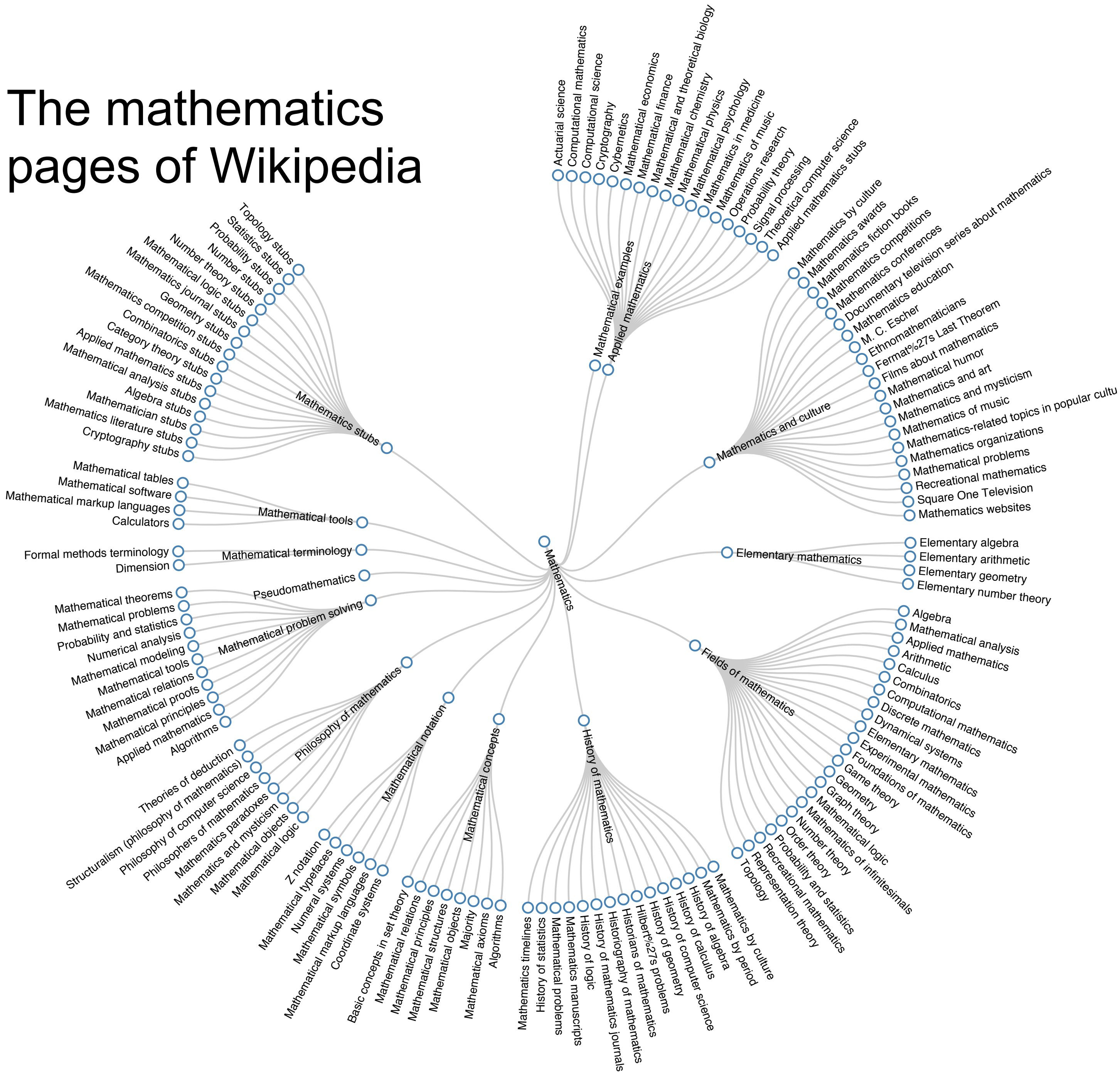
# *Quantum graphs*

- Vertices are connected by edges, on which $-\psi'' + V(x)\psi = E\psi$.
- The solutions are continuous and connected at the vertices by conditions such as

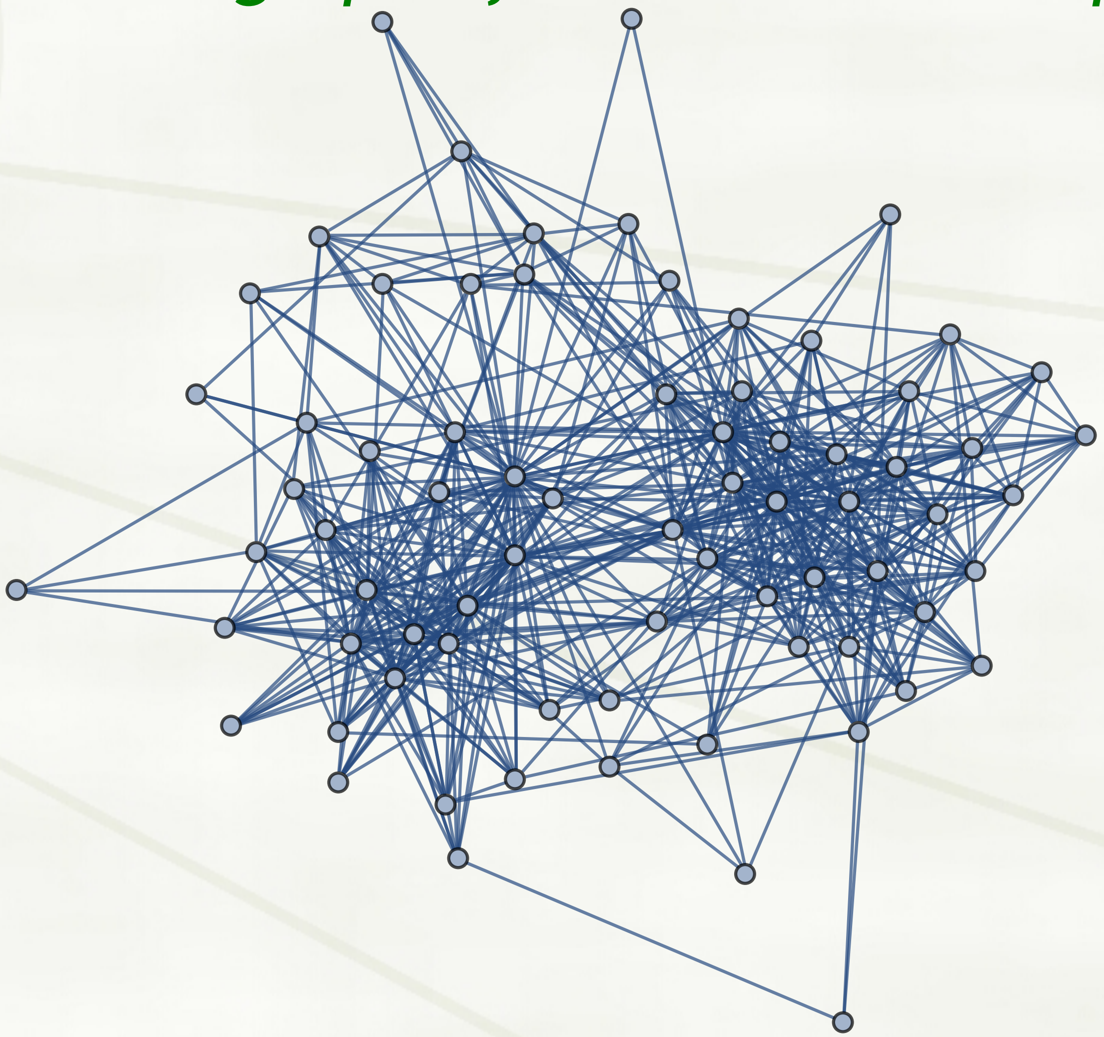
$$\sum_{e \sim v} f_e'(v^+) = 0.$$

"Kirchhoff conditions"

# The mathematics pages of Wikipedia



# The graph of PDE sites in wikipedia



*The adjacency matrix for Graph Theory*

For more, see
http://wikigraph.gatech.edu/

# *Quantum graphs*

- Kirchhoff conditions correspond to the energy form

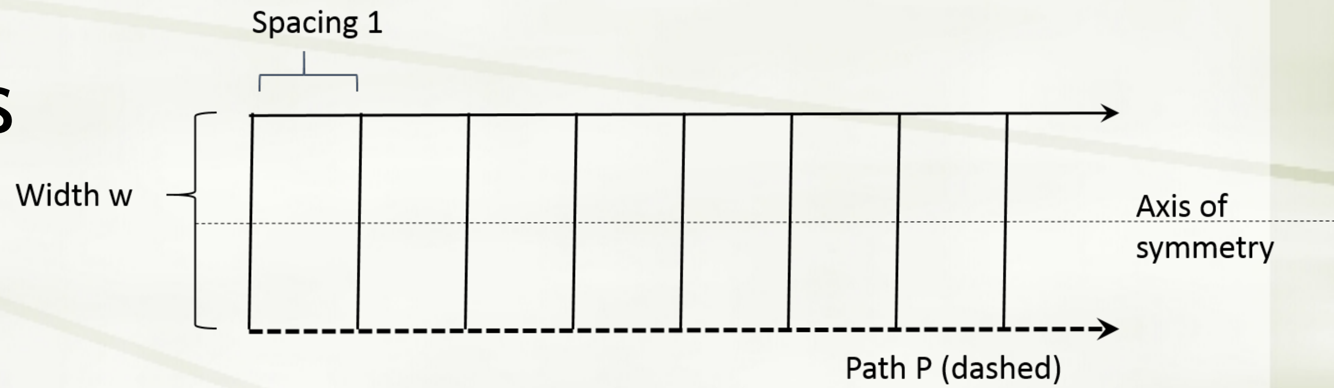
$$\phi \to \sum_e \int_e \left(|\phi'|^2 + V(x)|\phi|^2\right) dx$$

  on $H^1(\Gamma)$. (Like Neumann BC)

- If the graph tends to $\infty$, how well localized are the eigenfunctions?

# *Examples*

- Ladders




- Let V=0, E=-1 (outside a finite region). There is a symmetric solution that looks like $e^{-x}$ on the sides and constant on the rungs, and an antisymmetric one of the form $ge^{-|\ln\lambda_-|x}$ where g is periodic and $|\ln\lambda_-| > 1$.

# *Examples*

- Trees

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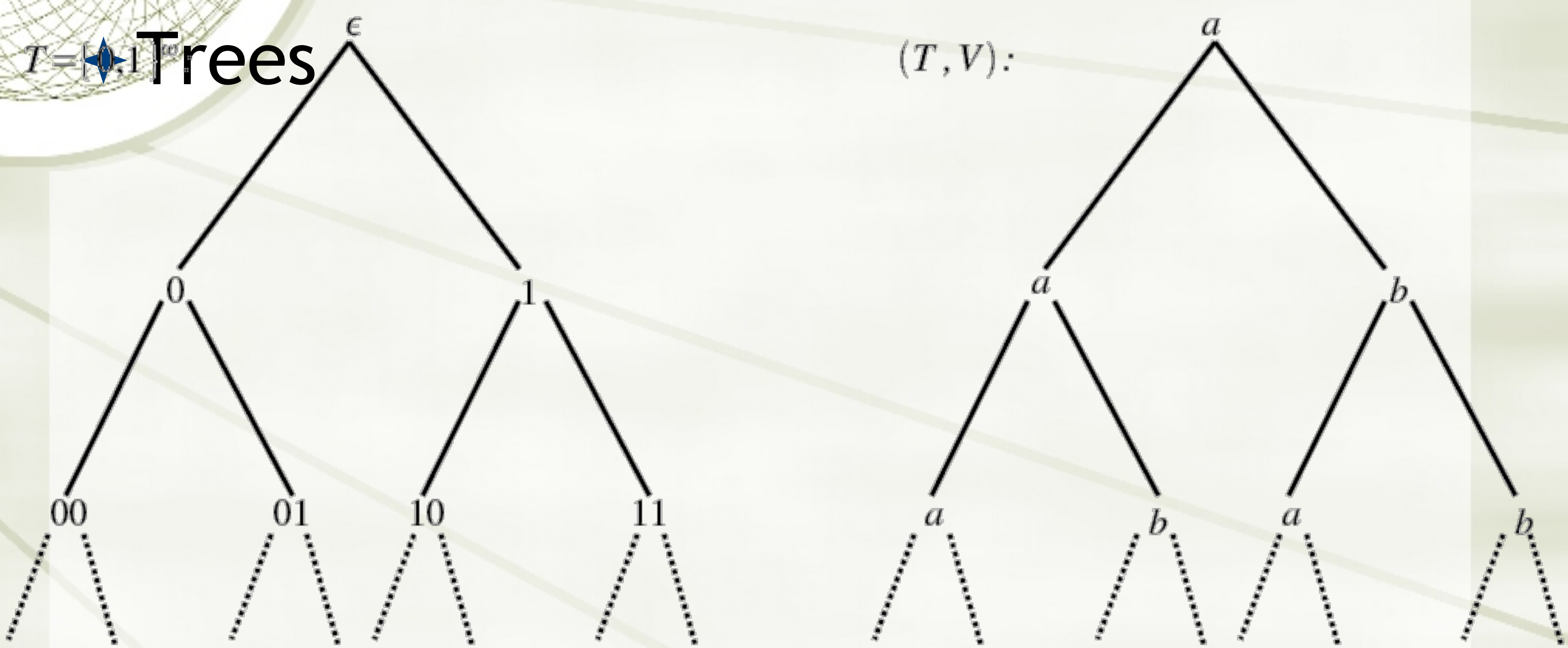

# Examples

- Trees
  - With branching number b and length L, the transfer matrix for the regular tree has smaller eigenvalue

$$= \frac{1}{\left(\frac{b}{2}+\frac{1}{2}\right)\cosh kL + \sqrt{\left(\left(\frac{b}{2}+\frac{1}{2}\right)\cosh kL\right)^2 - b}}$$

$$< \frac{1}{b\cosh kL}$$

(Here, $E = k^2$.)

# Examples

- Trees
  - We also work out the example of a 2-lengths regular tree.

(Here, $E = k^2$.)

# *Examples*

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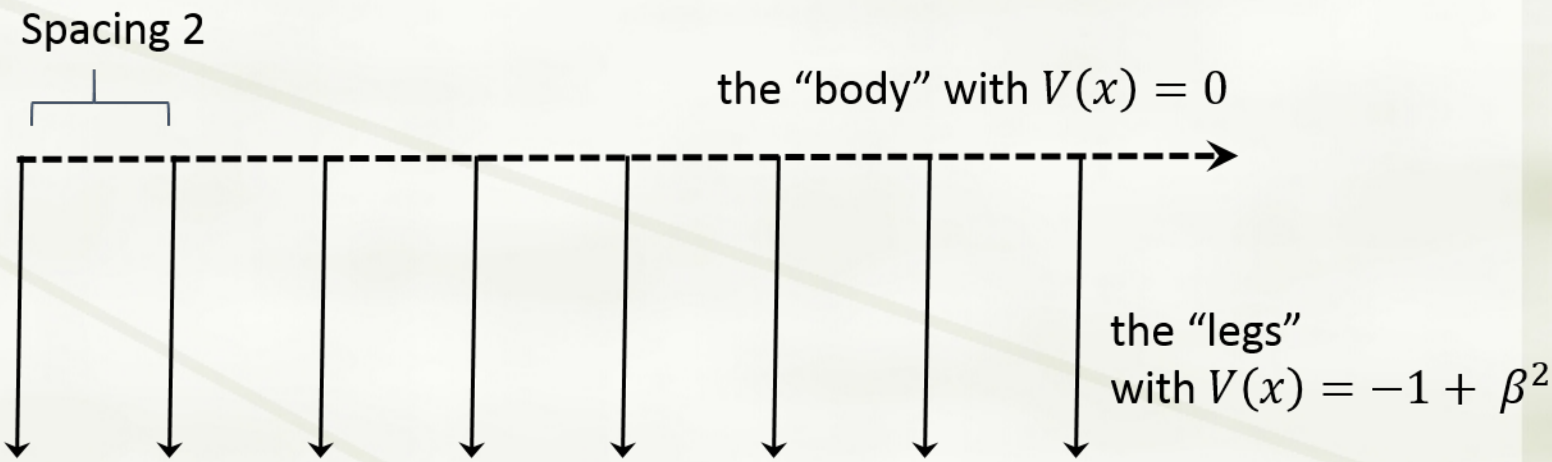
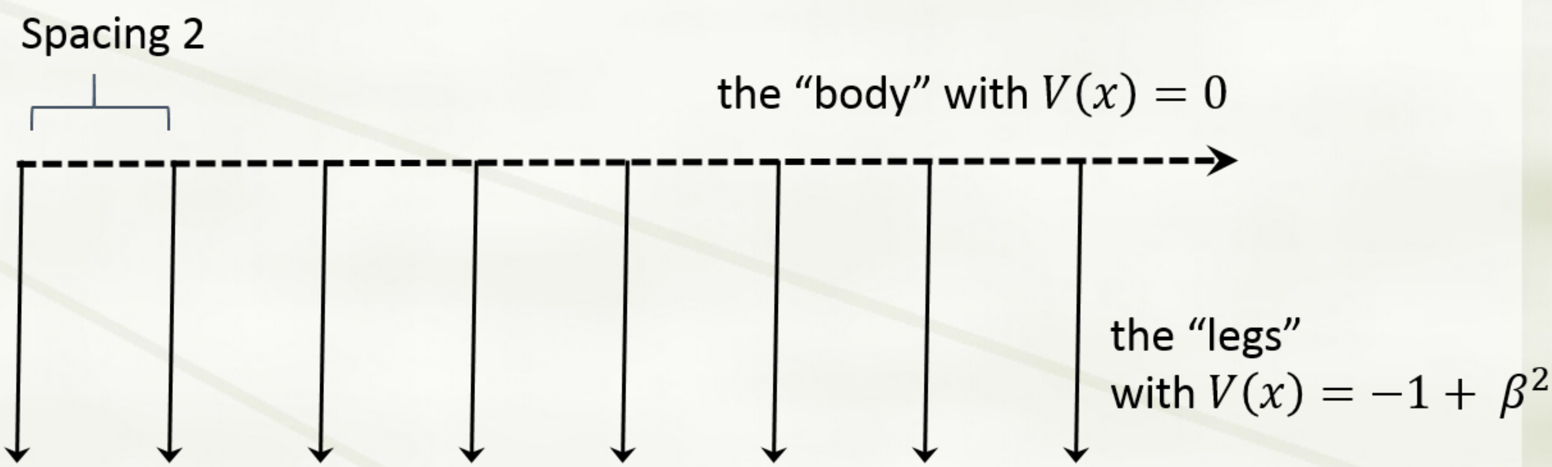


# *Examples*

- Millipedes

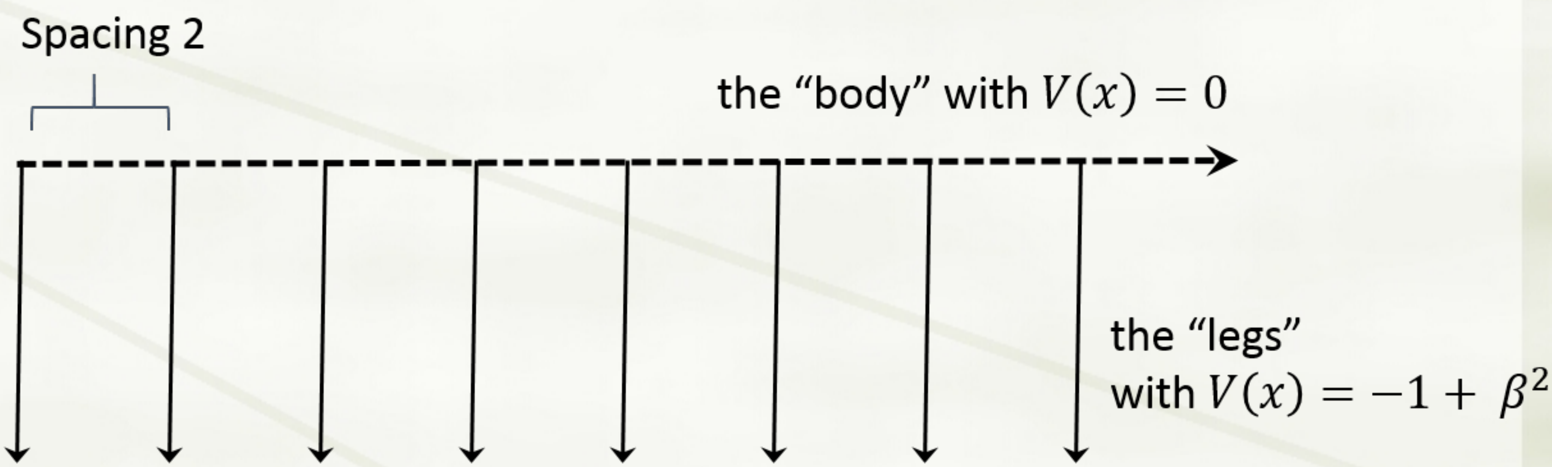



$$\lambda_- = e^{-2}\left(1 - \frac{\beta}{2}\right) + 0(\beta^2)$$

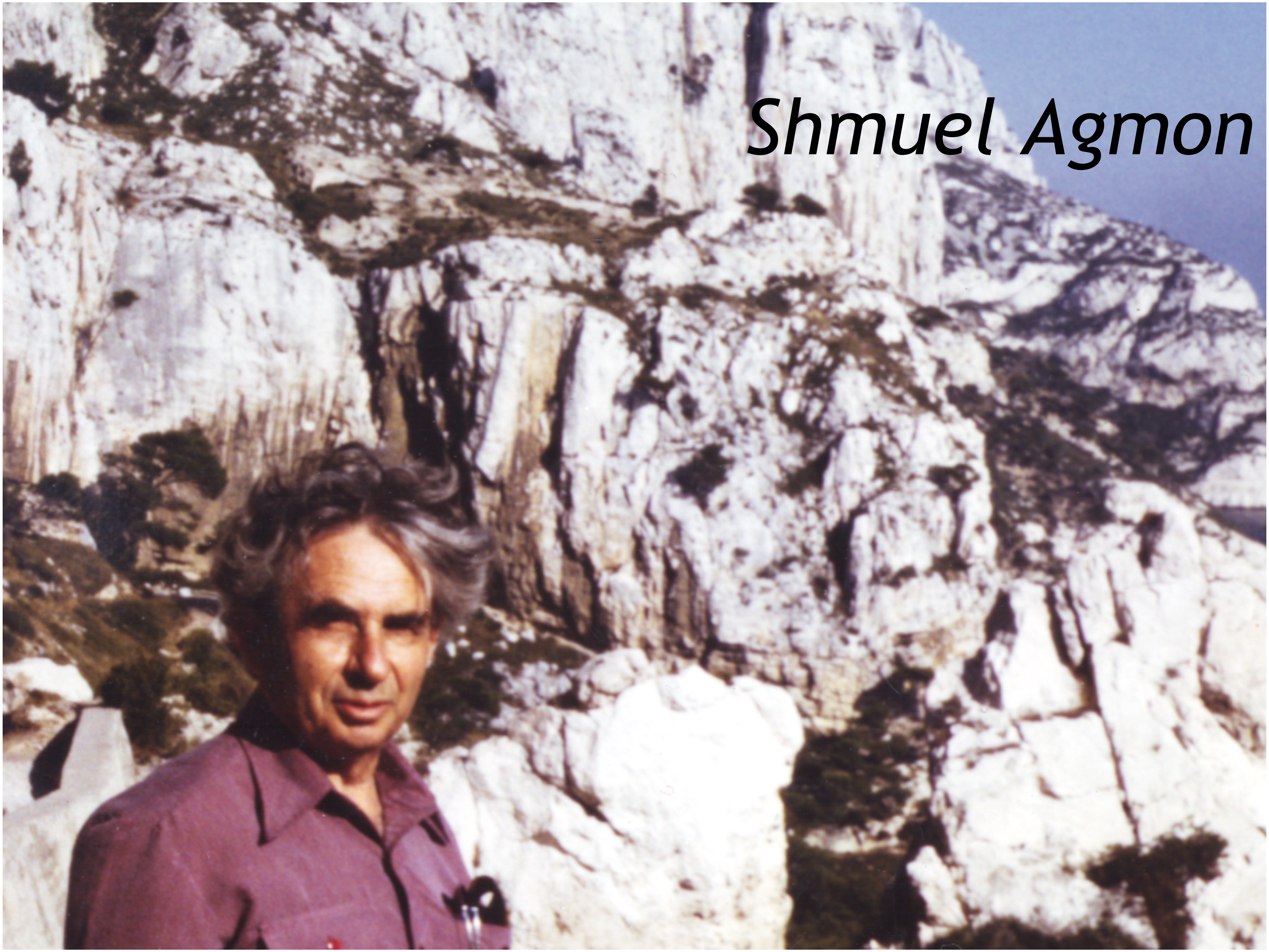
$$= e^{-2-\frac{\beta}{2}+0(\beta^2)}.$$

# Examples

- Millipedes

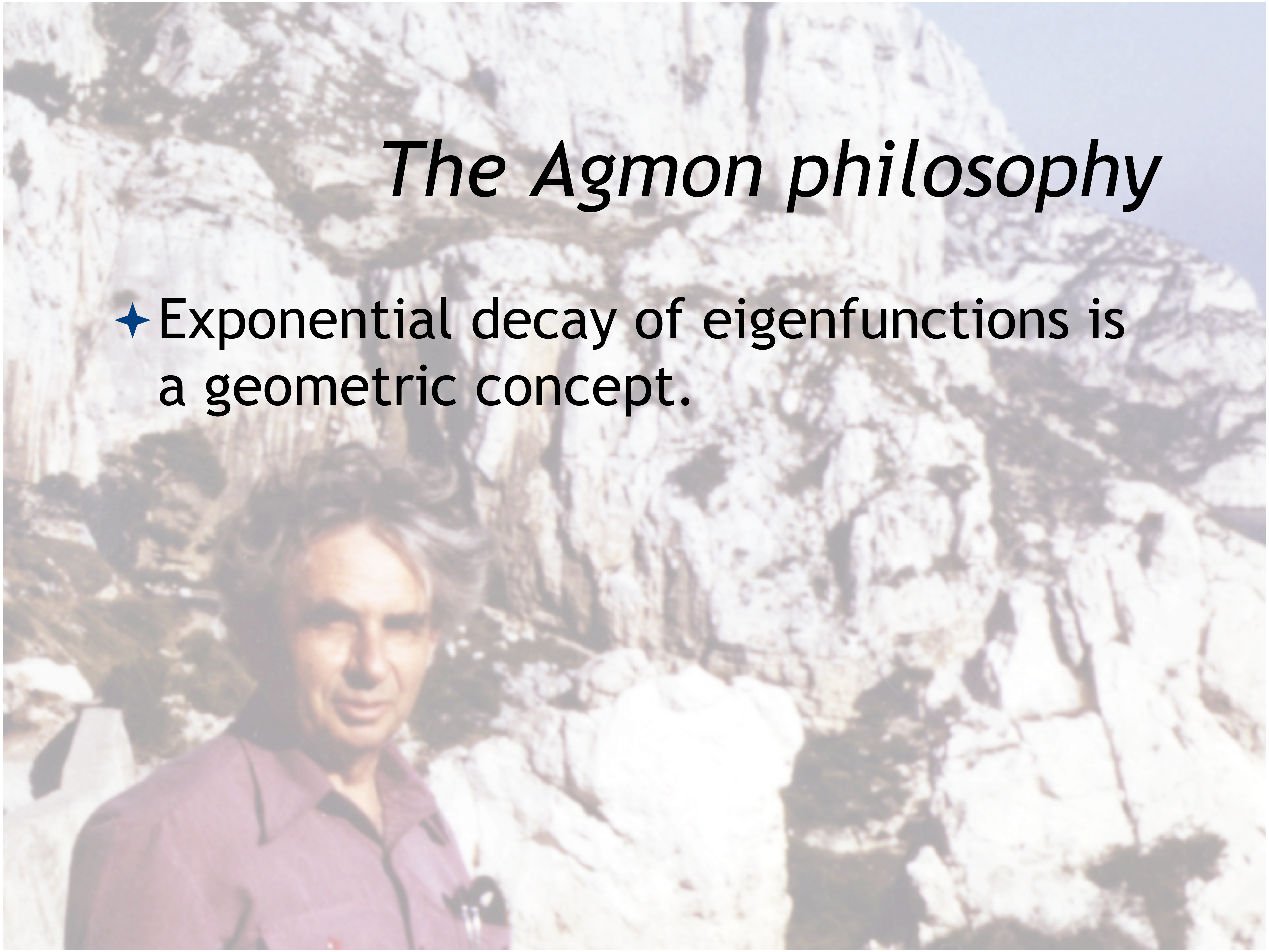


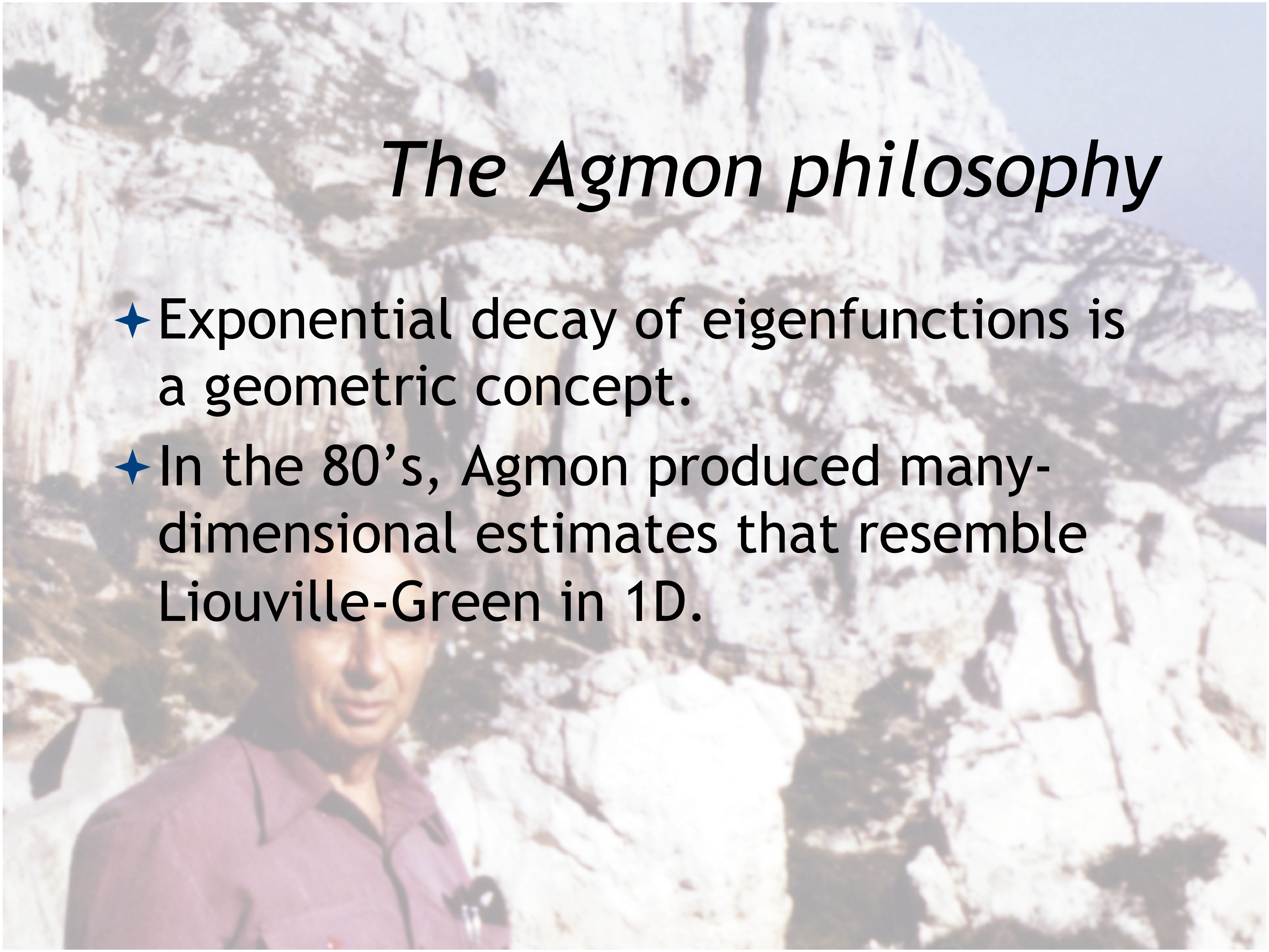

The spectral problem is equivalent to a problem on a half line, with delta potentials at regular intervals.


Shmuel Agmon

# *The Agmon philosophy*

- Exponential decay of eigenfunctions is a geometric concept.





# *The Agmon philosophy*

- Exponential decay of eigenfunctions is a geometric concept.
- In the 80’s, Agmon produced many-dimensional estimates that resemble Liouville-Green in 1D.

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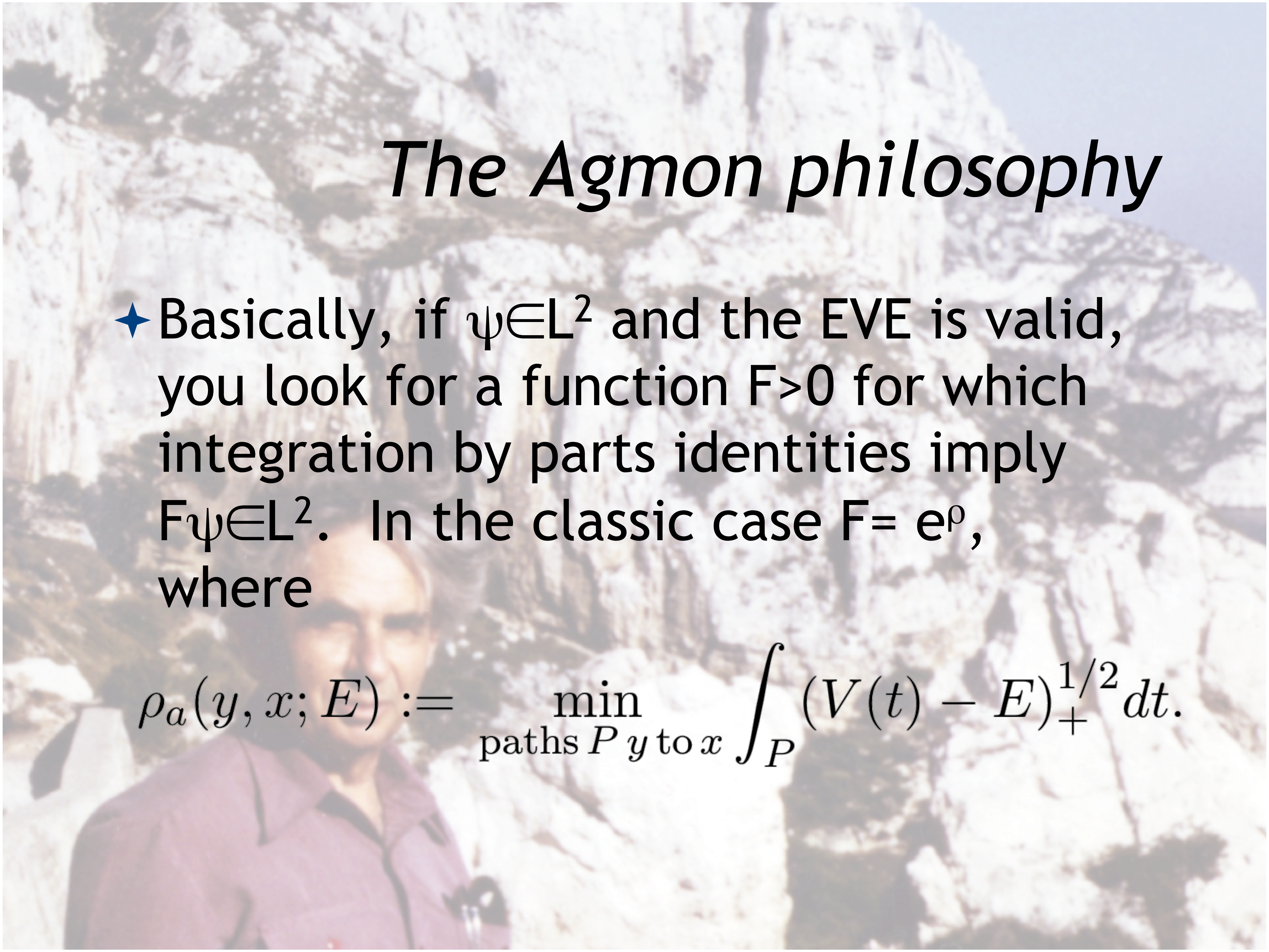
- Exponential decay of eigenfunctions is a geometric concept.
- Exponential dichotomy – in the nonoscillatory regime one expects asymptotic behavior like

  $$\exp(\pm\rho(x)).$$

- The Agmon metric depends on the potential ***and, as we shall show, the graph structure***.



# *The Agmon philosophy*

- Basically, if $\psi \in L^2$ and the EVE is valid, you look for a function F>0 for which integration by parts identities imply $F\psi \in L^2$. In the classic case $F = e^{\rho}$, where

$$\rho_a(y, x; E) := \min_{\text{paths } P \; y \text{ to } x} \int_P (V(t) - E)_+^{1/2} dt.$$

# *Our results*

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**Theorem 1.1.** *Suppose that $\Gamma_0 \subset \Gamma$ is a connected, infinite subgraph on which $\liminf(V(x) - E) > 0$. If $\psi \in L^2(\Gamma) \cap \mathcal{K}(\Gamma_0)$ satisfies*

$$-\psi'' + V(x)\psi = E\psi$$

*on the edges of $\Gamma_0$, then for any $\delta < \liminf(V - E)$,*

$$e^{\rho_a(x;E-\delta)}\psi \in H^1(\Gamma_0) \cap L^\infty(\Gamma_0). \tag{9}$$

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# *Our results*

1. The "classical action" estimate for eigensolutions on the line is valid for graphs.
2. Along a path, a refined estimate is possible in terms of the "fractions of the derivative" $p_k$. (Here we need assumptions that imply that eigenfunctions decay without changing sign.)

# *Our results*

**Theorem 1.2.** *Suppose that $\Gamma_0 \subset \Gamma$ is a connected, infinite subgraph on which $\liminf(V(x) - E) > 0$ and that $\psi \in L^2(\Gamma) \cap \mathcal{K}(\Gamma_0)$ satisfies*

$$-\psi'' + V(x)\psi = E\psi$$

*on the edges of $\Gamma_0$ and $\psi' < 0$ outside of a set of compact support. Consider any infinite path $P \subset \Gamma_0$, on which the fraction of the derivative exiting from a vertex $v$ is designated $p_v$. Then for any $\delta < \liminf(V - E)$, $e^{\rho_P(x,E-\delta)}\psi \in L^2(P)$. That is,*

$$\sqrt{\prod_{v \in P} \frac{1}{p_v}} e^{\rho_a(x,E-\delta)}\psi \in L^2(P) \cap L^\infty(P).$$

# *Our results*

1. The “classical action” estimate for eigensolutions on the line is valid for graphs.
2. Along a path, a refined estimate is possible.
3. On sufficiently regular graphs, an averaged wave function must decay more rapidly than the classical-action estimate.

# Our results

**Theorem 1.3.** *Suppose that $\Psi$ is the averaged eigenfunction on a quantum graph with regular topology corresponding to a solution $\psi$ of (1), for which $\psi \in L^2(\Gamma) \cap \mathcal{K}$, and that for all $x$ such that $\mathrm{dist}(0,x) = y$, $V(x) \geq V_m(y)$, where $\liminf(V_m(y) - E) > 0$. Define*

$$F_{\text{ave}}(y,E) := \left( \prod_{j:v_j<y} \sqrt{\frac{b_j}{a_j}} \right) e^{\int_0^y \sqrt{V_m(t)-E}\,dt}. \tag{12}$$

*Then for each $0 < \delta < \liminf(V_m - E)$,*

$$F_{\text{ave}}(y, E-\delta)\Psi \in H^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+).$$

# *The proof is not difficult*


So I’ll lead you through it.

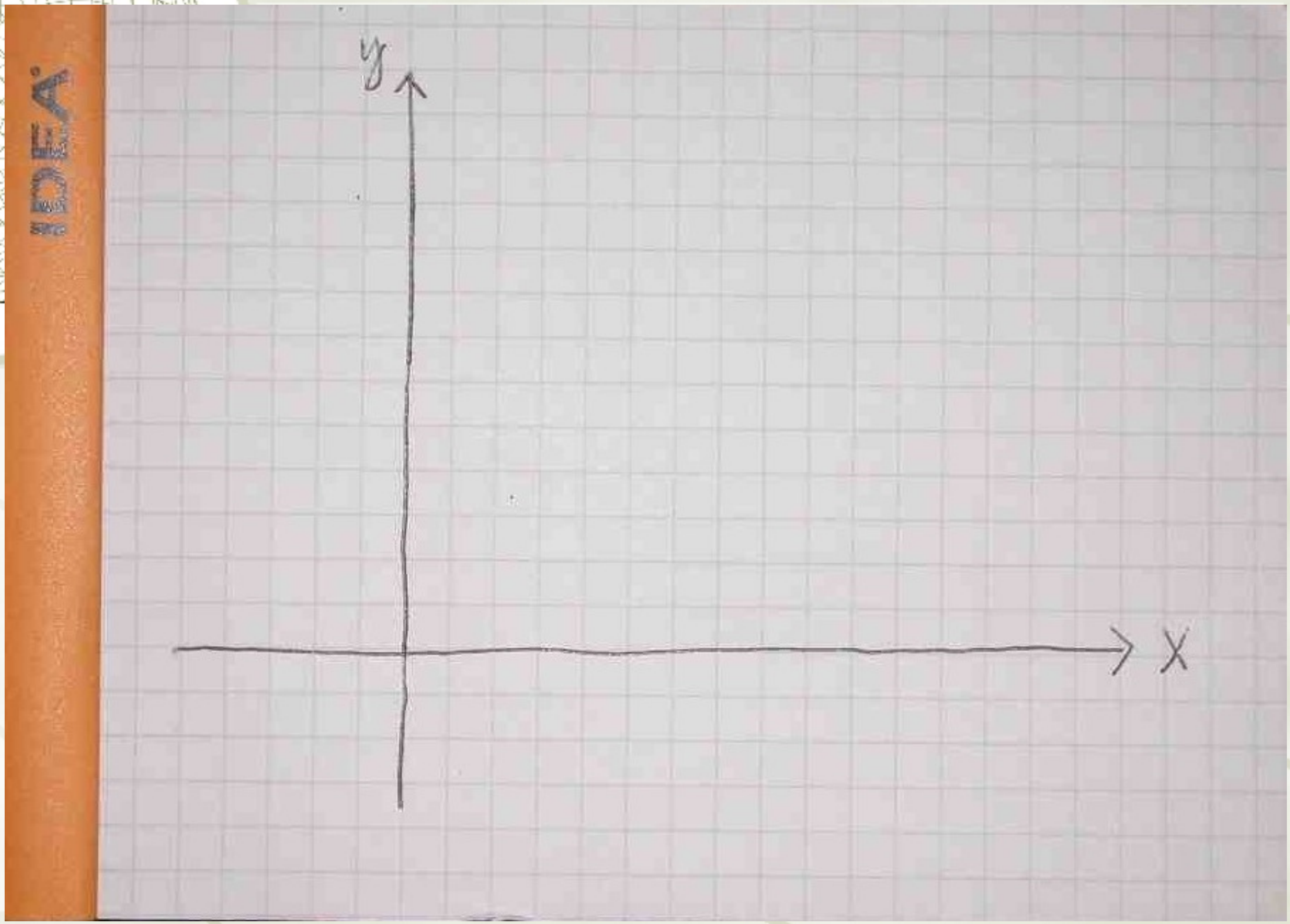
# Trick #1 – conjugation by F

**Lemma 3.1.** *Suppose that $\phi$ and $F > 0$ are real-valued functions on the metric graph $\Gamma$ such that $\phi \in AC^1$ and $F \in AC$. Then for any $x$ in an edge of $\Gamma$,*

$$(F\phi)'\left(\frac{\phi}{F}\right)' = (\phi')^2 - \left(\frac{F'}{F}\right)^2 \phi^2. \tag{14}$$

*Moreover, on any subgraph $\Gamma_0 \subset \Gamma$*

$$\sum_{e\in\Gamma_0}\int_e F\phi\left(-\frac{d^2}{dx^2} + V(x) - E\right)\frac{1}{F}\phi\,dx = -\sum_{v\in\Gamma_0}\sum_{e\in\Gamma_0, e\sim v} F\phi\frac{d}{dx}\left[\frac{1}{F}\phi\right](v^+)$$
$$+\sum_{e\in\Gamma_0}\int_e |\phi'|^2 + \left(V - E - \left|\frac{F'}{F}\right|^2\right)|\phi|^2 dx. \tag{15}$$



y
X

# *Trick #1 – conjugation by F*

- This means that if $V - E - |F'/F|^2 > \delta$, and we choose F to eliminate vertex terms, then the Sobolev norm of $\phi$ on the subgraph is controlled by

$$\sum_{e \in \Gamma_0} \int_e F\phi \left( -\frac{d^2}{dx^2} + V(x) - E \right) \frac{1}{F} \phi \, dx$$

# *Tricks #2 and 3 - cut off and exploit the EVE.*

- We chose $\phi/F = \psi$ outside a compact region, and after a calculation,

$$F^2\eta\psi\left(-\frac{d^2}{dx^2}+V(x)-E\right)\eta\psi \le C_0\chi_{\mathrm{supp}\,\eta}(x)(\psi)^2(x)+G'(x),$$

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Now add

$$\sum_{e \in \Gamma_0} \int_e (1 - \eta^2) \left[\left((F\psi)'\right)^2 + (F\psi)^2\right] dx,$$

# *Tricks #2 and 3 – cut off and exploit the EVE.*

- This means that the Sobolev norm of $F\psi$ outside a compact region is dominated by the $L^2$ norm of $\psi$ (even just on some compact subset).

# $L^2$ and $L^\infty$ estimates

$$
\begin{aligned}
|\phi(x_0)| = |\chi(x_0)\phi(x_0)| &= \left| \int_{x_0 - \frac{\ell_{\min}}{2}}^{x_0} (\chi(y)\phi(y))' dy \right| \\
&= \left| \int_{x_0 - \frac{\ell_{\min}}{2}}^{x_0} \left(\chi'(y)\phi(y) + \chi(y)\phi'(y)\right) dy \right| \\
&\leq \frac{1}{2} \int_{x_0 - \frac{\ell_{\min}}{2}}^{x_0} \left((\chi')^2(y) + (\phi(y))^2 + (\chi(y))^2 + (\phi'(y))^2\right) dy,
\end{aligned}
$$

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2. The millipede has decay faster than the classical-action bound, and our path-dependent estimate captures that.
3. The regular tree shows that the averaged bound is sharp. (Even one with two lengths.)

# The End