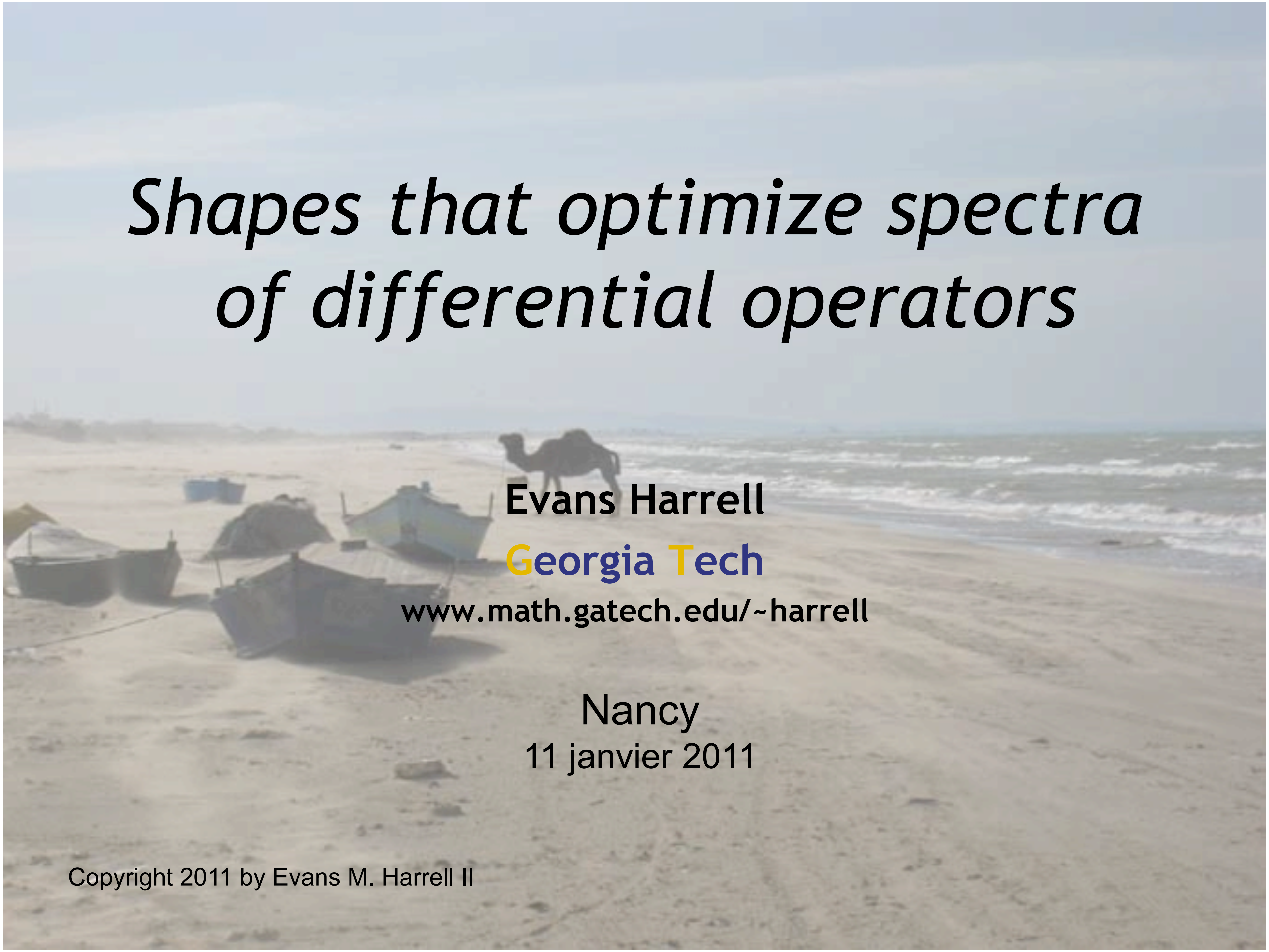


Shapes that optimize spectra of differential operators

Evans Harrell

Georgia Tech

www.math.gatech.edu/~harrell

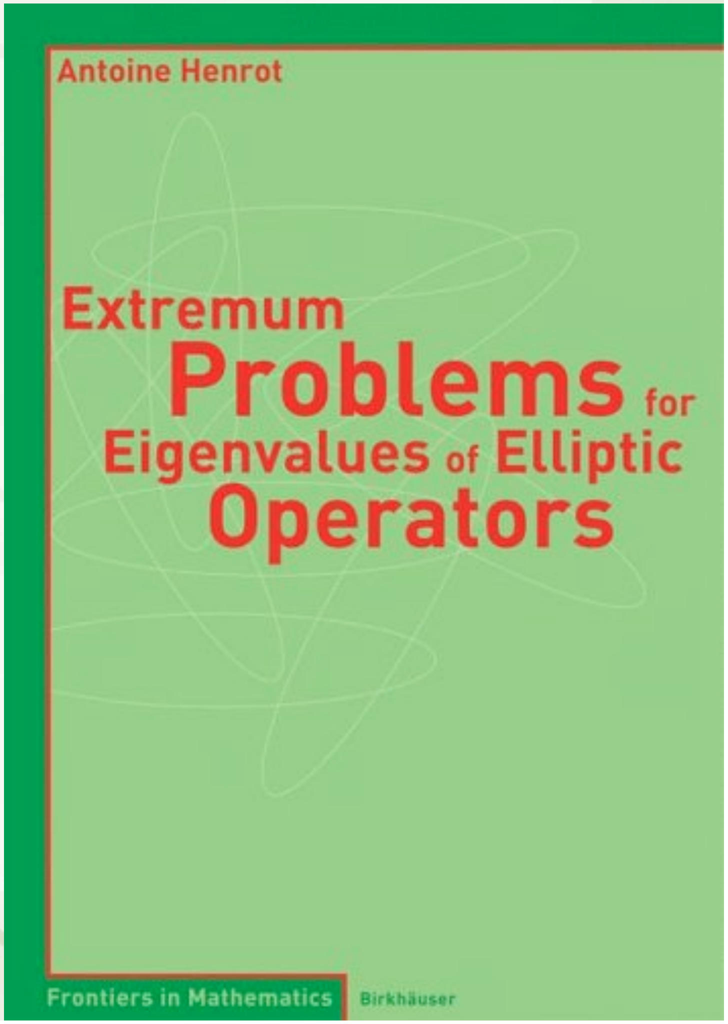
Nancy

11 janvier 2011

Abstract

- We'll discuss how to optimize the eigenvalues of some differential equations that depend on the geometry of a curve or surface, typically the Laplace operator plus a potential energy that is quadratic in curvature. I'll discuss the connections between the analytic tools and the geometry, and will present some sharp theorems, some illustrative examples, and open conjectures.

To extremists, things tend to look simple…



$-\Delta_{LB} + q(\kappa)$

- Here Δ is the Laplace-Beltrami operator on an immersed manifold and $q(\kappa)$ is an expression quadratic in the curvature.

$$-\ \Delta_{LB} + q(\kappa)$$

- Here Δ is the Laplace-Beltrami operator on an immersed manifold and $q(\kappa)$ is an expression quadratic in the curvature.

Why would anyone be interested in this operator?

On a (hyper) surface, what object is most like the Laplacian?

(Δ = the good old flat scalar Laplacian of Laplace)

Answer #1 (Beltrami's answer): Consider only tangential variations.

At a fixed point, orient Cartesian x_0 with the normal, then calculate

$$\sum_{j=1}^{d} \frac{\partial^2}{\partial {x_j}^2}$$

Difficulty:

- The Laplace-Beltrami operator is an intrinsic object, and as such is unaware that the surface is immersed!

- **Answer #2 (The nanoanswer):**

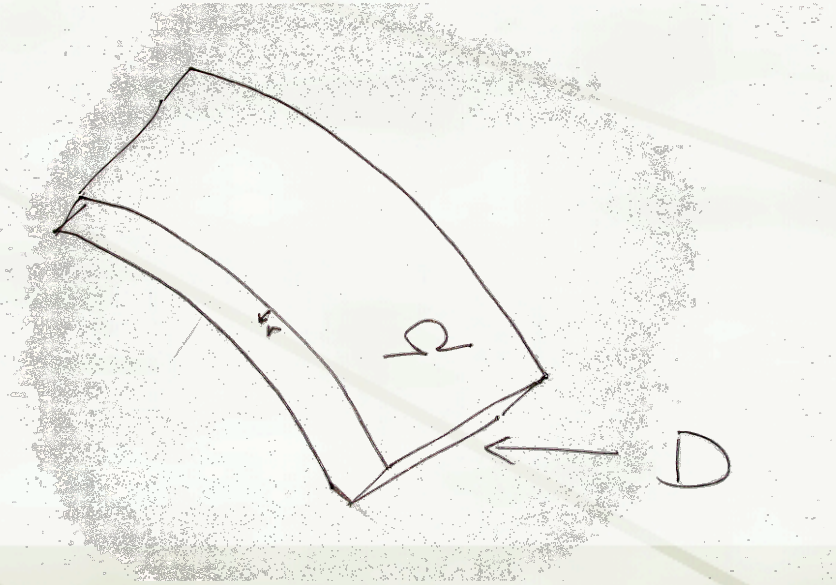
$$-\Delta_{LB} + q$$

- **Consider an electron or EM wave in a thin waveguide, as the width tends to zero.**
- **Since Da Costa, PRA, 1981: Perform a singular limit and renormalization to attain the surface as the limit of a thin domain.**

Thin domain of fixed width
variable r= distance from edge

Energy form in separated variables:

$$\int_D |\nabla_{||}\,\zeta|^2\, d^{d+1}x + \int_D |\zeta_r|^2\, d^{d+1}x$$



The effective potential when the Dirichlet Laplacian is squeezed onto a submanifold

$$- \Delta_{LB} + q,$$

$$q(\mathbf{x}) = \frac{1}{4}\left(\sum_{j=1}^{d} \kappa_j\right)^2 - \frac{1}{2}\sum_{j=1}^{d} \kappa_j^2$$

d=1, $q = -\kappa^2/4 \leq 0$ d=2, $q = -(\kappa_1-\kappa_2)^2/4 \leq 0$

Heisenberg's Answer

(if he had thought about it)

$$q(\mathbf{x}) = \frac{1}{4}\left(\sum_{j=1}^{d} \kappa_j\right)^2 .$$

Heisenberg's Answer

(if he had thought about it)

$$q(\mathbf{x}) = \frac{1}{4}\left(\sum_{j=1}^{d} \kappa_j\right)^2.$$

Note: q(**x**) ≥ 0 !

An "isoperimetric theorem," for λ_2.

Eigenfunctions of a self-adjoint operator, with different eigenvalues, are orthogonal. Therefore if we search over φ orthogonally to u_1,

$$\lambda_2 \leq \langle \varphi, A\,\varphi \rangle / ||\varphi||^2.$$

An "isoperimetric theorem," for λ_2.

Eigenfunctions of a self-adjoint operator, with different eigenvalues, are orthogonal. Therefore if we search over φ orthogonally to u_1,

$$\lambda_2 \leq <\varphi, A\,\varphi>/||\varphi||^2.$$

Problem: We don't know u_1 *a priori*. One way around this is a lemma of J. Hersch:

An "isoperimetric theorem," for λ_2.

Lemma. *(J. Hersch). Let Ω be a two-dimensional, closed, smooth Riemannian manifold of the topological type of the sphere, and specify a bounded, positive, measurable function ρ on Ω. Then there exists a conformal transformation $\Phi : \Omega \to S^2 \subset R^3$, embedded in the standard way as the unit sphere, such that*

$$\int_{S^2} \mathbf{x}\rho(\Phi^{-1}(\mathbf{x}))Jd\hat{S} = \mathbf{0}. \tag{3}$$

Jacobian

An "isoperimetric theorem," for λ_2.

For the trial function φ let's choose one of the Cartesian coordinates x,y,z on S^2, but "pull back" to Ω with the inverse of Hersch's conformal transformation. Let the resulting functions on Ω be called X,Y,Z. What do we know about X,Y,Z?

An "isoperimetric theorem," for λ_2.

For the trial function φ let's choose one of the Cartesian coordinates x,y,z on S^2, but "pull back" to Ω with the inverse of Hersch's conformal transformation. Let the resulting functions on Ω be called X,Y,Z. What do we know about X,Y,Z?

1. The functions X,Y,Z are orthogonal, because the functions x,y,z are orthogonal on S^2.
   * Note: The restrictions of x,y,z to S^2 are the spherical harmonics = eigenfunctions:

     $- \nabla^2 x = 2\,x,$

     $- \nabla^2 y = 2\,y,$

     $- \nabla^2 z = 2\,z,$

An "isoperimetric theorem," for λ_2.

For the trial function φ let's choose one of the Cartesian coordinates x,y,z on S^2, but "pull back" to Ω with the inverse of Hersch's conformal transformation. Let the resulting functions on Ω be called X,Y,Z. What do we know about X,Y,Z?

1. The functions X,Y,Z are orthogonal.

2. $X^2 + Y^2 + Z^2 = 1$, because $x^2 + y^2 + z^2 = 1$.

An "isoperimetric theorem," for λ_2.

For the trial function φ let's choose one of the Cartesian coordinates x,y,z on S^2, but "pull back" to Ω with the inverse of Hersch's conformal transformation. Let the resulting functions on Ω be called X,Y,Z. What do we know about X,Y,Z?

1. The functions X,Y,Z are orthogonal.
2. $X^2 + Y^2 + Z^2 = 1$, because $x^2 + y^2 + z^2 = 1$.
3. Identifying now ρ with u_1,

   $\langle X,u_1\rangle =$ Likewise for Y, Z.

$$\int_{S^2} \mathbf{x}\rho(\Phi^{-1}(\mathbf{x}))J d\hat{S} = \mathbf{0}.$$

Ready to roll with Rayleigh and Ritz:

Let's choose the trial function in

$$R(\zeta) := \frac{\int_\Omega |\nabla \zeta|^2 dS - \frac{1}{4} \int_\Omega (\kappa_2 - \kappa_1)^2 |\zeta|^2 dS}{\int_\Omega |\zeta|^2 dS}$$

as ζ =X, Y, or Z. Considering for example X, conformality implies that

$$\int_\Omega |\nabla X|^2 dS = \int_{S^2} |\nabla x|^2 d\hat{S} = \frac{8\pi}{3}$$

Ready to roll with Rayleigh and Ritz:

Observing that

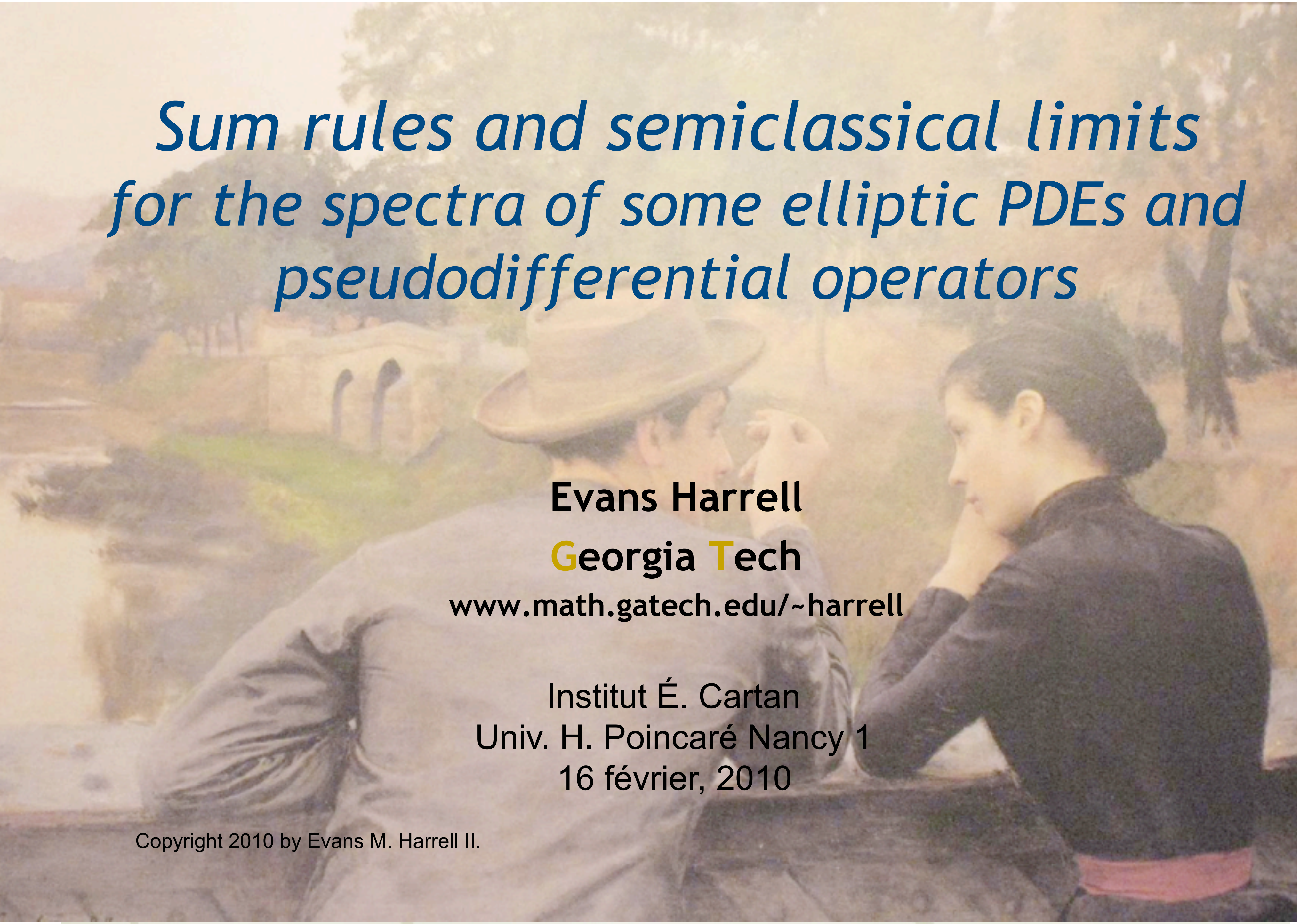
$$a \leq \frac{b_j}{c_j}$$

$\Rightarrow$

$$a \leq \frac{\sum_j b_j}{\sum_j c_j} :$$

$$\lambda_2 \leq \frac{8\pi - \int_\Omega (\kappa_2 - \kappa_1)^2 dS}{\int_\Omega 1 dS}.$$

Equality iff sphere. Why?



Sum rules and semiclassical limits for the spectra of some elliptic PDEs and pseudodifferential operators

Evans Harrell

Georgia Tech

www.math.gatech.edu/~harrell

Institut É. Cartan
Univ. H. Poincaré Nancy 1
16 février, 2010

Commutators of operators

- $[H, G] := HG - GH$
- $[H, G]\ \phi_k = (H - \lambda_k)\ G\ \phi_k$
- If $H=H^*$,

$$<\phi_j,[H, G]\ \phi_k> = (\lambda_j - \lambda_k)\ <\phi_j,G\phi_k>$$

1^{st} and 2^{nd} commutators *(H-Stubbe '97)*

$$\frac{1}{2}\sum_{\lambda_j \in J} (z-\lambda_j)^2 \langle [G,[H,G]]\phi_j, \phi_j\rangle - \sum_{\lambda_j \in J} (z-\lambda_j) \| [H,G]\phi_j \|^2$$

$$=$$

$$\sum_{\lambda_j \in J} \sum_{\lambda_k \in J^c} (z-\lambda_j)(z-\lambda_k)(\lambda_k-\lambda_j) | \langle G\phi_j, \phi_k\rangle |^2$$

The only assumptions are that H and G are self-adjoint, and that the eigenfunctions are a complete orthonormal sequence. (If continuous spectrum, need a spectral integral on right.)

1^{st} and 2^{nd} commutators *(H-Stubbe '97)*

$$\frac{1}{2}\sum_{\lambda_j \in J} (z-\lambda_j)^2 \left\langle [G,[H,G]]\phi_j, \phi_j \right\rangle - \sum_{\lambda_j \in J} (z-\lambda_j) \|[H,G]\phi_j\|^2$$

$$=$$

$$\sum_{\lambda_j \in J} \sum_{\lambda_k \in J^c} (z-\lambda_j)(z-\lambda_k)(\lambda_k - \lambda_j) |\left\langle G\phi_j, \phi_k \right\rangle|^2$$

If $J = \{\lambda_1, ..., \lambda_n\}$ and $z \in (\lambda_n, \lambda_{n+1})$, the right side ≤ 0. If $H = -\Delta_{LB} + V(x)$, and G is a Cartesian coordinate of the ambient space, the commutators can be calculated in terms of curvature (indep. of V, which might or might not = $q(\kappa)$).

Commutators: [A,B] := AB-BA

3a. The equations of space curves are commutators:

$$\left[\frac{d}{ds}, \mathbf{x}\right] = \mathbf{t}$$

$$\left[\frac{d}{ds}, \mathbf{t}\right] = \kappa\mathbf{n}$$

Note: curvature is defined by a second commutator

The Serret-Frenet equations as commutator relations:

$$[H, X_m] = -\frac{d^2X_m}{ds^2} - 2\frac{dX_m}{ds}\frac{d}{ds} = -\kappa n_m - 2t_m\frac{d}{ds}, \qquad (2.2)$$

$$[X_m[H, X_m]] = 2t_m^2. \qquad (2.3)$$

1^{st} and 2^{nd} commutators for Schrödinger on immersed mflds

Let M be a smooth immersed curve. Then for $\varphi \in W_0^1(M)$,

$$\sum_{m=0}^{d} \|[H, X_m]\,\varphi\|^2 = 4\int_M \left(\left|\frac{d\varphi}{ds}\right|^2 + \frac{\kappa^2}{4}|\varphi|^2 \right) ds.$$

1st and 2nd commutators for Schrödinger on immersed mflds

$$\sum_k (z-\lambda_k)_+^2 \le \frac{4}{d}\sum_k (z-\lambda_k)_+ \left(\int_M \left(|\nabla\phi_k|^2 + \frac{h^2}{4}|\phi_k|^2\right) dV\right)$$

A quadratic “Yang-type inequality” implies numerous universal bounds for sums and gaps of eigenvalues, partition functions, etc.

1^{st} and 2^{nd} commutators for Schrödinger on immersed mflds

$$\left(1+\frac{2}{d}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i+\frac{2}{d}\frac{1}{k}\sum_{i=1}^{k}\delta_i-\sqrt{D_{nk}}$$

$$\leq \lambda_{k+1}$$

$$\leq \left(1+\frac{2}{d}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i+\frac{2}{d}\frac{1}{k}\sum_{i=1}^{k}\delta_i+\sqrt{D_{dk}},$$

where D_{dk} depends only on the eigenvalues up through k and the dimension, and

$$\delta_i := \int_M \left(\frac{|h|^2}{4}-V\right)u_i^2.$$

A simplification, for intuitive purposes:

$$\lambda_{k+1} \leq \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{4}{n}\, \bar{\delta},$$

$$\delta := \sup_M \left(\tfrac{h^2}{4} - V\right).$$

The bounds on λ_{k+1} are attained for all k with $\lambda_{k+1} \neq \lambda_k$, when

1. The potential is of the form $g\, h^2$.

2. The submanifold is a sphere.

(For details see articles linked from my webpage beginning with Harrell-Stubbe Trans. AMS 349(1997)1797.)

How can these inequalities be universal, when Colin de Verdière has shown that any finite positive numbers can be Laplace spectra for a manifold?

How can these inequalities be universal, when Colin de Verdière has shown that any finite positive numbers can be Laplace spectra for a manifold?

- A: Conditions for immersibility.

Partition function

$$Z(t) := \mathrm{tr}(\exp(-tH)).$$

Corollary 4.5 a) *Let H be as (3.1), with M a compact, smooth submanifold. Then $t^{\frac{d}{2}} \exp(-\delta t)\, Z(t)$ is a nondecreasing function;*

b) *For H_g be of the form (1.10) on a smooth, compact submanifold M, $t^{\frac{d}{2\sigma}} Z(t)$ is a nondecreasing function.*

Theorem 3.1 *Let $\overline{M}$ be $\mathbb{S}^m$ or $\mathbb{F}P^m$ and let $X : M \longrightarrow \overline{M}$ be an isometric immersion of mean curvature h. For any bounded potential q on M, the spectrum of $H = -\Delta_g + q$ (with Dirichlet boundary conditions if $\partial M \neq \emptyset$) must satisfy, $\forall k \in \mathbb{N}$, $k \geq 1$,*

$$(I)\ n\sum_1^k(\lambda_{k+1}-\lambda_i)^2 \leq 4\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\left(\lambda_i+\bar{\delta}_i\right),$$

where $\bar{\delta}_i := \frac{1}{4}\int_M(|h|^2+c(n)-4q)u_i^2$,

$$(II)\ \lambda_{k+1} \leq \left(1+\frac{2}{n}\right)\frac{1}{k}\sum_{i=1}^k\lambda_i + \frac{2}{n}\frac{1}{k}\sum_{i=1}^k\bar{\delta}_i + \sqrt{\bar{D}_{nk}}$$

where

$$\bar{D}_{nk} := \left(\left(1+\frac{2}{n}\right)\frac{1}{k}\sum_1^k\lambda_i + \frac{2}{n}\frac{1}{k}\sum_{i=1}^k\bar{\delta}_i\right)^2 - \left(1+\frac{4}{n}\right)\frac{1}{k}\sum_1^k\lambda_i^2 - \frac{4}{n}\frac{1}{k}\sum_{i=1}^k\lambda_i\bar{\delta}_i \geq 0,$$

A lower bound is also possible along the lines of Theorem 2.1. As in the previous section, the following simplifications follow easily:

Corollary 3.1 *With the notation of Theorem 3.1 one has, $\forall k \geq 1$,*

$$\lambda_{k+1} \leq \left(1+\frac{4}{n}\right)\frac{1}{k}\sum_{i=1}^k\lambda_i + \frac{4}{n}\,\bar{\delta},$$

where $\bar{\delta} := \frac{1}{4}\sup\left(|h|^2+c(n)-4q\right)$.

Extension of Reilly's inequality *(with El Soufi-Ilias, TAMS 2010)*

$$\lambda_k \leq C(d,k)\|h\|_\infty^2$$

$$C_R = \frac{(d+4)^2}{d^2(d+2)}k^{2/d} - \frac{4}{d^2}$$

An "isoperimetric theorem," for λ_2.

- Consider the thin-domain operator on a closed, simply connected surface in R^3,

$$-\nabla^2 - (\kappa_2 - \kappa_1)^2/4.$$

- The ground state is maximized (at 0) by the sphere. Let's fix the area and ask after the maximum of the second eigenvalue.

One dimension - a loopy problem

The foregoing operators can be caricatured with a family of one-dimensional Schrödinger operators on a closed loop, of the form:

$$\mathbf{H(g)} = -\frac{\mathbf{d^2}}{\mathbf{ds^2}} + \mathbf{g}\kappa^{\mathbf{2}}$$

where s=arclength, κ is the curvature, and g is a real "coupling constant." The length is fixed. What shapes optimize low-lying eigenvalues, gaps, etc., and for which values of g?

Optimizers of λ_1 for loops

- $g < 0$. Not hard to see λ_1 uniquely *max*imized by circle. No minimizer - a kink corresponds to a negative multiple of δ^2 (yikes!).
- $g > 1$. No maximizer. A redoubled interval can be thought of as a singular minimizer.
- $0 < g \leq 1/4$. Exner-Harrell-Loss '99: circle is minimizer. Conjectured that the bifurcation was at $g = 1$. (When $g=1$, if the length is 2π, both the circle and the redoubled interval have $\lambda_1 = 1$.)
- If the embedding in R^m is neglected, the bifurcation is at $g = 1/4$ (Freitas, CMP 2001).

It is rather easy to see that the fundamental eigenvalue is maximized by the circle for all nonnegative g.

$$\mathbf{H(g) = -\frac{d^2}{ds^2} + g\kappa^2}$$

$\mathcal{C}$ - length 2π, closed

$$-\frac{d^2}{ds^2} + g\,K^2 \quad , \quad g \leq 0$$

$\Rightarrow$ $\lambda_1 \leq 0$, maximized by circle.

Proof Rayleigh-Ritz $\Rightarrow$

$$\lambda_1 \int 1^2 ds \leq \int ((1')^2 + g K^2 1^2) ds$$

$$2\pi \lambda_1 = -|g| \int K^2 ds = -|g| \frac{\int K^2 ds \int 1^2 ds}{2\pi}$$

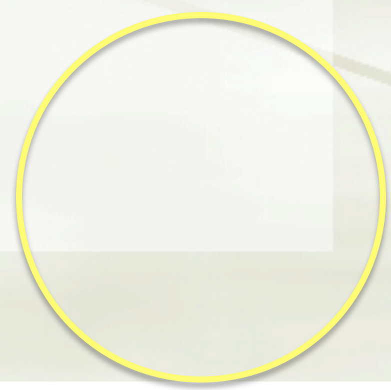
Cauchy-Schwarz $\leq -\frac{|g|}{2\pi} \left(\int K \cdot 1\right)^2 = -|g| \cdot 2\pi$

$$\lambda_1 \leq -|g|$$

Equality iff $K = \text{cst.}$ a.e. (circle).

With the other sign, the fundamental eigenvalue is minimized by the circle for $0 < g < \frac{1}{4}$ (Exner-Harrell-Loss Conf. Proc. 1999).

$$\mathbf{H(g) = -\frac{d^2}{ds^2} + g\kappa^2}$$



With the other sign, the fundamental eigenvalue is minimized by the circle for $0 < g < 1/4$ (Exner-Harrell-Loss Conf. Proc. 1999).

$$\mathbf{H(g) = -\frac{d^2}{ds^2} + g\kappa^2}$$

The conjecture is that there is a bifurcation at g=1, below which the circle is always the optimizer. (Remains open, some progress by Linde, Proc. AMS 2006.)

"The loop problem"

$$\mathbf{H(g) = -\frac{d^2}{ds^2} + g\kappa^2}$$

If $0 < g \leq 1/4$, the unique curve minimizing λ_1 is the circle (Exner-Harrell-Loss '99). If $g > 1$, no longer true.

**What happens in between? OPEN.

Minimality when g ≤ 1/4.

Proof. a) Assume first that $0 < g < 1/4$. The minimal value of λ_1, which we denote λ_*, is

$$\inf_{\kappa} \inf_{\zeta} \int \left(\left(\frac{\mathrm{d}\zeta}{\mathrm{d}s} \right)^2 + g\, \kappa^2 \zeta^2 \right) \mathrm{d}s,$$

Because the quantity in question is an iterated infimum, it may be calculated in the other order. By Cauchy-Schwarz's inequality

$$2\pi = \int \frac{\kappa}{\zeta}\zeta \mathrm{d}s \le \left(\int \frac{1}{\zeta^2}\mathrm{d}s\right)^{1/2}\left(\int \kappa^2\zeta^2\mathrm{d}s\right)^{1/2},$$

with equality only if

$$\kappa = \left(2\pi / \int \frac{1}{\zeta^2}\mathrm{d}s\right)\frac{1}{\zeta^2}.$$

A nonquadratic functional

$$E(\zeta) := \int \left(\frac{\mathrm{d}\zeta}{\mathrm{d}s}\right)^2 \mathrm{d}s + \frac{4\pi^2 g}{\int \left(\frac{1}{\zeta^2}\right) \mathrm{d}s}.$$

A nonquadratic functional

$$E(\zeta) \; := \; \int \left(\frac{\mathrm{d}\zeta}{\mathrm{d}s}\right)^2 \mathrm{d}s \; + \; \frac{4\pi^2 g}{\int \left(\frac{1}{\zeta^2}\right) \mathrm{d}s}.$$

$$\lambda_* \leq E(1) = 4g\pi^2 \leq \pi^2 \quad \text{for } g \leq \tfrac{1}{4}.$$

Lemma 5: *If $E(\zeta) \leq \pi^2$ for a positive test function ζ normalized in L^2, then*

$$\inf_s (\zeta(\mathrm{s})) > 1 - \frac{\sqrt{E(\zeta)}}{\pi}.$$

Proof of Lemma 5.

$$E(\zeta) > \int_0^1 (\zeta')^2 \, \mathrm{d}s = \int_0^1 (\zeta - \zeta_{\min})'^2 \mathrm{d}s \geq \pi^2 \int_0^1 (\zeta - \zeta_{\min})^2 \mathrm{d}s,$$

Minimizer therefore exists. Its Euler equation is

$$-\zeta_*'' + M \frac{1}{\zeta_*^3} = C \zeta_*,$$

$$M = \frac{4\pi^2 g}{\left(\int_0^1 \frac{1}{\zeta_*^2} \, ds\right)^2}$$

Solution of Euler equation of the form:

$$\zeta_*^2 = 1 + \sqrt{1 - M/\lambda_*}\ \cos\left(2\sqrt{\lambda_*}(s - s_0)\right)$$

Nonconstant solution of this form excluded because

$\lambda_* < \pi^2$.

Current state of the loop problem

- Benguria-Loss, *Contemp. Math.* 2004. Exhibited a one-parameter continuous family of curves with $\lambda_1 = 1$ when $g = 1$. It contains the redoubled interval and the circle.
- B-L also showed that an affirmative answer is equivalent to a standing conjecture about a sharp Lieb-Thirring constant.

Current state of the loop problem

- Burchard-Thomas, J. Geom. Analysis 15 (2005) 543. The Benguria-Loss curves are local minimizers of λ_1.
- Linde, Proc. AMS **134** (2006) 3629. Conjecture proved under an additional geometric condition. L raised general lower bound to 0.6085.
- AIM Workshop, Palo Alto, May, 2006.

Another loopy equivalence

- Another equivalence to a problem connecting geometry and Fourier series in a classical way:
  - Rewrite the energy form in the following way:

$$E(u) := \int_0^{2\pi} \left(|u'|^2 + \kappa^2 |u|^2 \right) ds = \int_0^{2\pi} \left| \frac{d\left(e^{i\theta(s)} u(s)\right)}{ds} \right|^2 ds$$

  - Is

$$E(u) \geq \int_0^{2\pi} u^2 \quad ?$$

Another loopy equivalence

- Replace s by z = exp(i s) and regard the map

  z $\rightarrow$ *w* := u exp(i θ)

  as a map on C that sends the unit circle to a simple closed curve with winding number one with respect to the origin. Side condition that the mean of *w*/|w| is 0.

- For such curves, is $\|w'\| \geq \|w\|$?

Loop geometry and Fourier series

- In the Fourier (= Laurent) representation,

$$w = \sum_{k > -\infty}^{\infty} c_k z^k$$

the conjecture is that if the mean of w/|w| is 0, then:

$$\sum_k k^2 |c_k|^2 \geq \sum_k |c_k|^2$$

Or, equivalently,

$$|c_0|^2 \leq \sum_{|k| \geq 2} (k^2 - 1) |c_k|^2$$

Another isoperimetric theorem for

$$\mathbf{H(g) = -\frac{d^2}{ds^2} + g\kappa^2}$$

- The eigenvalue λ_2 of H(-1) is uniquely maximized, and = 0, when the loop is the circle. (Harrell-Loss '98)

A nice $q(\kappa)$ for immersed manifolds is $g\times$ the square of mean curvature from the Laplace-Beltrami operator on an immersed closed manifold in any D. If $g=-1$, he second eigenvalue is still maximized by the sphere.

- In 2D, conformal mapping allows a proof for $-1 \leq g < 0$, even for a certain family of negative-definite quadratic forms $q(\kappa)$.

A nice $q(\kappa)$ for immersed manifolds is $g\times$ the square of mean curvature from the Laplace-Beltrami operator on an immersed closed manifold in any D. If $g=-1$, he second eigenvalue is still maximized by the sphere.

- El Soufi has shown the same for $-1 < g < 0$, in dimension > 2 (Indiana UMJ, 2009)
- Dimension 1 for this range of g remains open!

THE

END