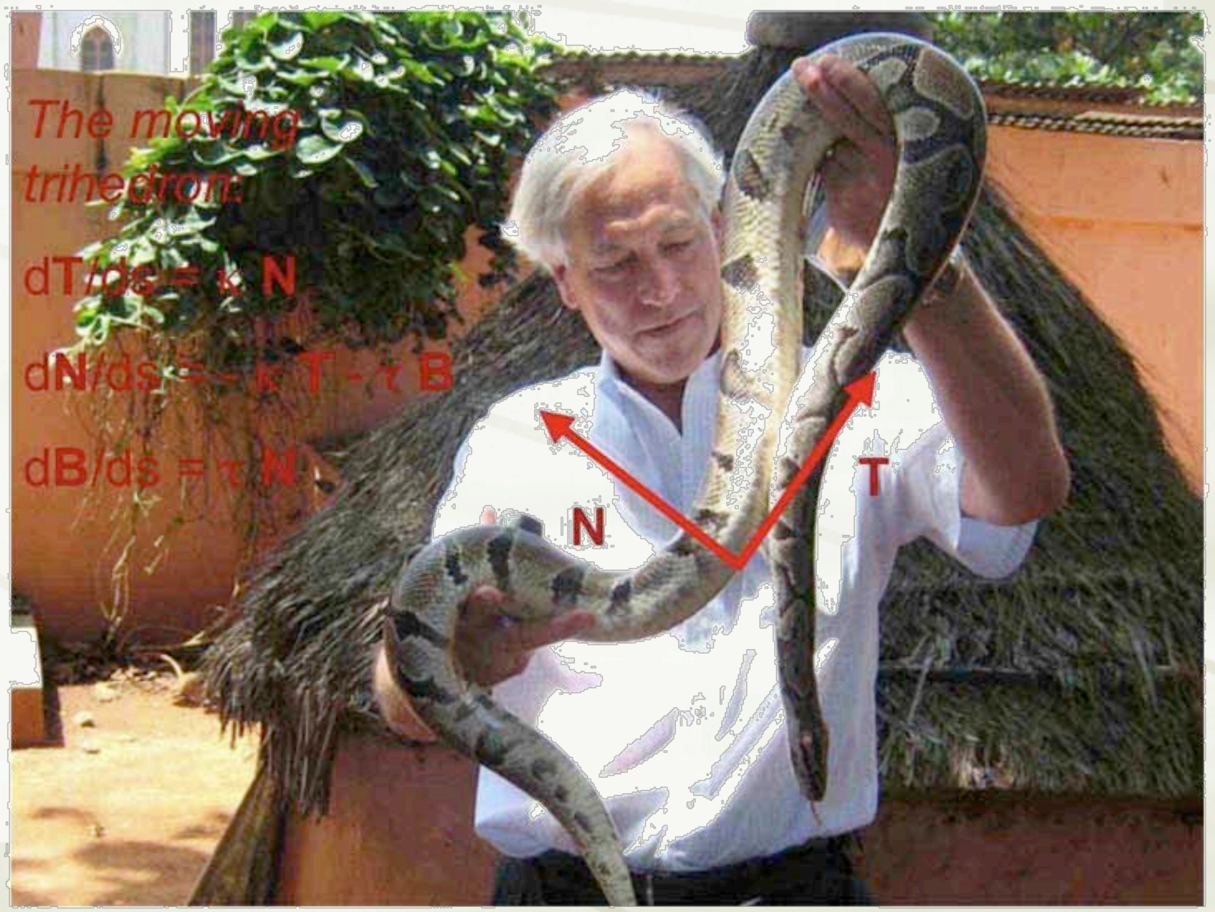
Universal patterns in spectra of differential operators

This lecture will not involve snakes, machetes, or other dangerous objects.



Laplace, Laplace-Beltrami, and Schrödinger

$$T = -\Delta = -\nabla^2 := -\sum_{\alpha=1}^{d} \frac{\partial^2}{\partial x_\alpha^2}$$

$$\langle \phi, T\phi \rangle = \int_\Omega |\nabla \phi|^2$$

$$H = T + V(x)$$

Laplace, Laplace-Beltrami, and Schrödinger

$$H\, u_k = \lambda_k\, u_k$$

Musicians' vibrating membranes

Inuit

Musicians' vibrating membranes

Inuit

Musicians' vibrating membranes

Inuit

Musicians' vibrating membranes

Inuit

West Africa

Musicians' vibrating membranes

Inuit

West Africa

USA

Physicists' vibrating membranes

Sapoval 1989

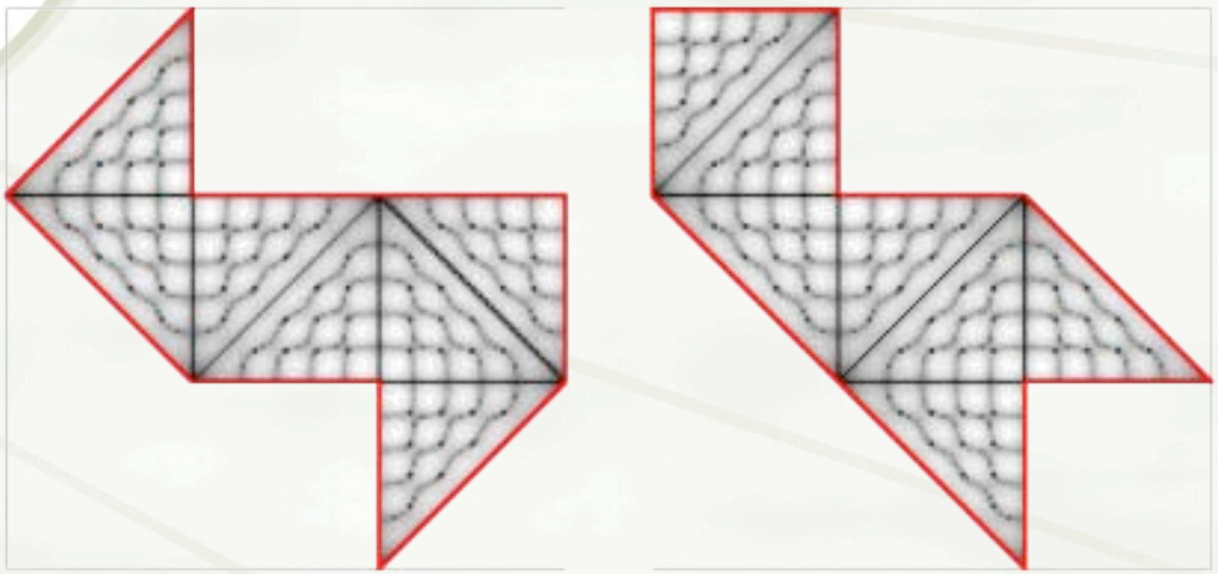
Physicists' vibrating membranes

Sapoval 1989

Physicists' vibrating membranes

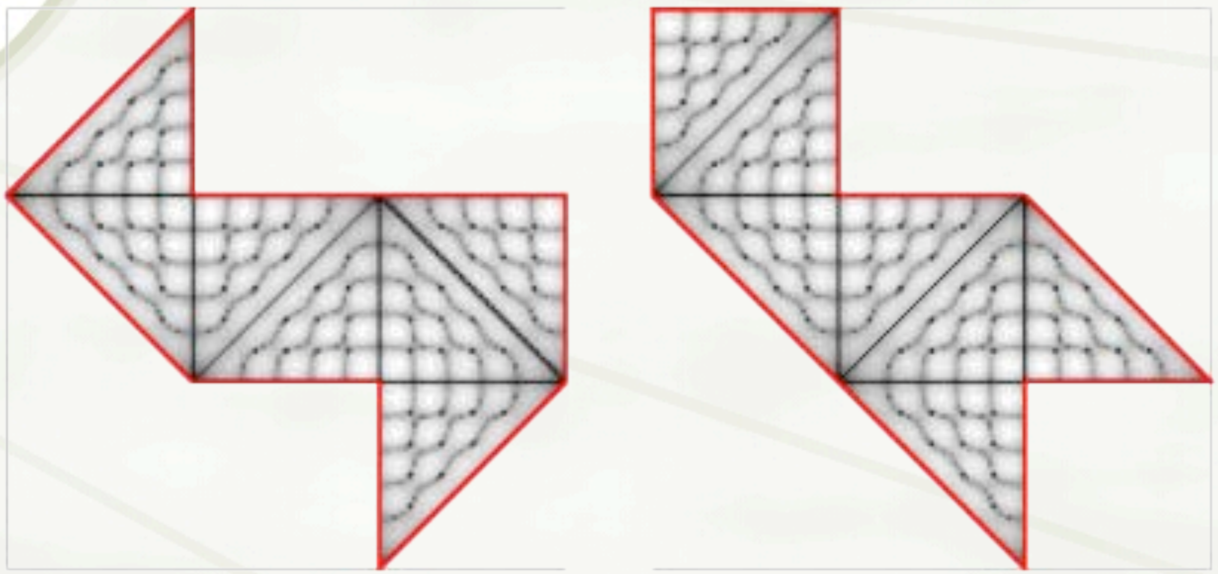
Sapoval 1989

Mathematicians' vibrating membranes



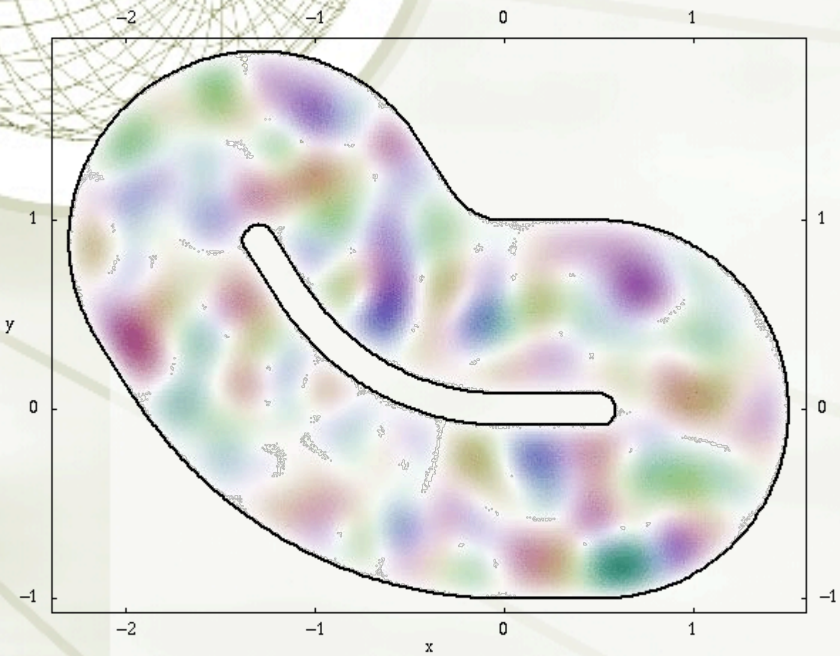
Wolfram MathWorld, following Gordon, Webb, Wolpert. ("Bilby and Hawk")

Mathematicians' vibrating membranes

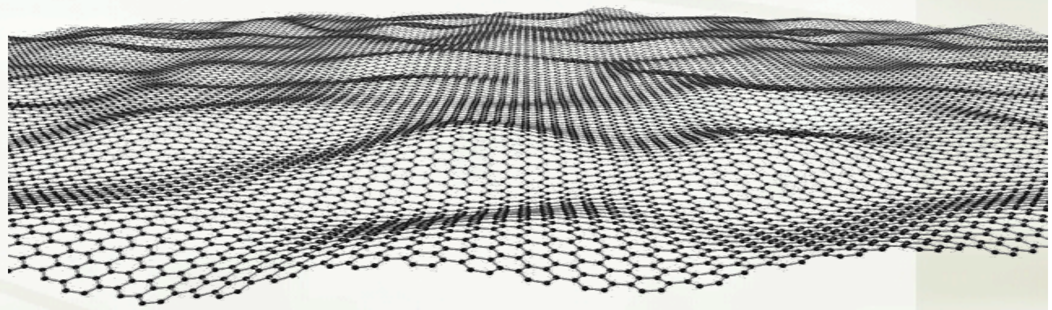


Wolfram MathWorld, following Gordon, Webb, Wolpert. ("Bilby and Hawk")

Quantum waveguides - same math (sort of), different physics



Ljubljana



Max Planck Inst.

Laplace eigenvalues - domains in R^d

n												
1	1	1	1	1	52.638	1	5.6313	1	5.7855	1	5.784	1
2	1.438	1.43	1.404	1.435	122.82	2.3333	7.1815	1.2753	30.484	5.2689	14.684	2.5387
3	2.04	2.027	1.961	2.037	210.55	4	12.791	2.2713	74.917	12.949	14.684	2.5387
4	2.571	2.548	2.442	2.577	228.1	4.3333	13.094	2.3251	139.1	24.042	26.378	4.5605
5	2.854	2.823	2.861	2.86	333.37	6.3333	17.068	3.0309	223.02	38.548	26.378	4.5605
6	3.623	3.57	3.506	3.633	368.47	7	18.854	3.348	326.7	56.468	30.47	5.268
7	4.184	4.153	3.924	4.18	473.74	9	19.851	3.5251	450.12	77.8	40.704	7.0373
8	4.554	4.507	4.452	4.557	491.29	9.3333	24.18	4.2939	593.28	102.55	40.704	7.0373
9	4.861	4.811	4.582	4.866	543.92	10.333	27.538	4.8901	756.2	130.71	49.224	8.5104
10	5.15	5.095	5.279	5.148			30.01	5.3291	938.86	162.28	49.224	8.5104
11				5.646								
12				6.27								
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16				8.247								
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23				10.83								
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25				11.697								
26												
27												
28												
29												
	---											
	d=2			d=2	d=2		d=2		d=2		unit disc	
	Even-Pieransky 1999			Sridhar-	Cureton-Kuttler		Driscoll 1997		Grinfeld Strang			
	tangrams			tangram	equ. Triangle		tangram 2		128-gon			

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	d=2 Even-Pieransky 1999 tangrams			d=2 Sridhar- tangram	d=2 Cureton-Kuttler equ. Triangle		d=2 Driscoll 1997 tangram 2		d=2 Grinfeld Strang 128-gon		unit disc	

There's something Strang about this column.

Should you believe this man?

For the 128-gon,

$\lambda_1 = 5.78552$	$\lambda_6 = 326.69528$
$\lambda_2 = 30.48357$	$\lambda_7 = 450.11529$
$\lambda_3 = 74.91726$	$\lambda_8 = 593.28245$
$\lambda_4 = 139.09646$	$\lambda_9 = 756.19675$
$\lambda_5 = 223.02237$	$\lambda_{10} = 938.85822$

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But according to the Ashbaugh-Benguria Theorem,

$\lambda_2 / \lambda_1 \leq 2.5387\ldots!$

(In 2 D)

Table (24) shows our estimates for the first ten simple eigenvalues on the regular 128-sided polygon.

Aha!

Universal spectral patterns

- The top of the spectrum is controlled by the bottom!
- Example: Two classic results relate the spectrum to the volume of the region. For definiteness, consider

$$H = -\nabla^2 \text{ , Dirichlet BC}$$

(i.e., functions vanish on $\partial\Omega$).

A simple universal spectral pattern (for the Dirichlet Laplacian)

- The Weyl law (B=std ball):

  $\lambda_k(\Omega) \sim C_d\,(|B|k\,/\,|\Omega|)^{2/d}$ as $k \to \infty$.
- Faber-Krahn:

  $\lambda_1(\Omega) \geq \lambda_1(B)\,(|B|\,/\,|\Omega|)^{2/d}$.
- Therefore, *universally*,

  $\lim\,(\lambda_k/\lambda_1\,k^{2/d}) \leq C_d/\lambda_1(B)$.
- Equality only for the ball.

Statistics of spectra

For Laplacian on a bounded domain,

$$\left(\left(1+\frac{2}{d}\right)\overline{\lambda_k}\right)^2-\left(1+\frac{4}{d}\right)\overline{\lambda_k^2}\geq 0$$

Harrell-Stubbe TAMS 1996

Statistics of spectra

$$\left(\left(1+\frac{2}{d}\right)\overline{\lambda_k}\right)^2 - \left(1+\frac{4}{d}\right)\overline{\lambda_k^2} \geq 0.$$

A reverse Cauchy inequality:

The variance is dominated by the square of the mean.

Statistics of spectra

$$D_k := \left(\left(1 + \tfrac{2}{d}\right) \overline{\lambda_k}\right)^2 - \left(1 + \tfrac{4}{d}\right) \overline{\lambda_k^2} \geq 0.$$

$$\lambda_{k+1} \leq \left(1 + \frac{2}{d}\right) \overline{\lambda_k} + \sqrt{D_k}.$$

$$\lambda_{k+1} - \lambda_k \leq 2\sqrt{D_k}$$

Means of ratios:

Corollary 3.1 *For $k \geq j^{\frac{1+\frac{d}{2}}{1+\frac{d}{4}}}$, the means of the eigenvalues of the Dirichlet Laplacian satisfy a universal Weyl-type bound,*

$$\overline{\lambda_k}/\overline{\lambda_j} \leq 2\left(\frac{1+\frac{d}{4}}{1+\frac{d}{2}}\right)^{1+\frac{2}{d}}\left(\frac{k}{j}\right)^{\frac{2}{d}}. \tag{3.4}$$

Harrell-Hermi preprint 2007

Reverse arithmetic-geometric mean inequalities:

For Laplacian on a bounded domain,
$0 < r \leq 3$,

$$\overline{\lambda_k^r}^{\frac{1}{r}} \leq \frac{d\ e^{2/d}}{d+2r} \left(\prod_{j=1}^{k} \lambda_j \right)^{\frac{1}{k}}$$

Harrell-Stubbe in prep.

Surfaces and other "immersed manifolds"

Theorem 2.1 *Let $X : M \longrightarrow \mathbb{R}^m$ be an isometric immersion. We denote by h the mean curvature vector field of X (i.e the trace of its second fundamental form). For any bounded potential q on M, the spectrum of $H = -\Delta + q$ (with Dirichlet boundary conditions if $\partial M \neq \emptyset$) must satisfy, $\forall k \geq 1$,*

$$(I)\ n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\,(\lambda_i + \delta_i)$$

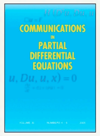
El Soufi-Harrell-Ilias preprint 2007

Surfaces and other “immersed manifolds”

For a Schrödinger operator T+V(x) on a closed hypersurface,

$$\lambda_2 - \lambda_1 \leq \frac{4}{d}\left\langle u_1, \left(-\Delta + \frac{h^2}{4}\right) u_1 \right\rangle.$$

Harrell CPDE 2007



Communications in Partial Differential Equations

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Commutators, Eigenvalue Gaps, and Mean Curvature in the Theory of Schrödinger Operators

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URL: http://dx.doi.org/10.1080/03605300500532889

Statistical bounds for eigenvalue gaps, which are *all sharp* in the case V = g h^2, standard sphere.

El Soufi-Harrell-Ilias preprint 2007

Surfaces and other "immersed manifolds"

$$(II\ a)\ \lambda_{k+1} \leq \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{4\delta}{n}.$$

$$\text{where } \delta := \sup\left(\frac{|h|^2}{4} - q\right).$$

For Laplace-Beltrami:

$$\lambda_k \leq C_R(n,k)\, \|h\|_\infty^2 \, .$$

El Soufi-Harrell-Ilias preprint 2007

Sum rules and Yang-type inequalities

$$1 \leq \frac{4}{dk} \sum_{j=1}^{k} \frac{\int_M \left(|\nabla u_i|^2 + \frac{|h|^2}{4} u_i^2 \right) dVol}{\lambda_{k+1} - \lambda_j}$$

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{d} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\int_M \left(|\nabla u_i|^2 + \frac{|h|^2}{4} u_i^2 \right) dVol \right)$$

"Universal" constraints on eigenvalues

- H. Weyl, 1910, Laplace operator $\lambda_n \sim n^{2/d}$
- W. Kuhn, F. Reiche, W. Thomas, 1925, sum rules for energy eigenvalues of Schrödinger operators
- L. Payne, G. Pólya, H. Weinberger, 1956: gaps between eigenvalues of Laplacian controlled by sums of lower eigenvalues:
- E. Lieb and W. Thirring, 1977, P. Li, S.T. Yau, 1983, sums of eigenvalues, powers of eigenvalues.
- Hile-Protter, 1980, stronger but more complicated analogue of PPW

"Universal" constraints on eigenvalues

- Ashbaugh-Benguria 1991, isoperimetric conjecture of PPW proved.
- H. Yang 1991-5, unpublished, complicated formulae like PPW, respecting Weyl asymptotics.
- Harrell, Harrell-Michel, Harrell-Stubbe, 1993-present, commutators.
- Hermi PhD thesis, articles by Levitin and Parnovsky.

PPW – the grandfather of many universal spectral patterns:

- All eigenvalues are controlled by the ones lower down! (Payne-Pólya-Weinberger, 1956)

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{d}\overline{\lambda_k} := \frac{4}{d}\,\frac{1}{k}\sum_{j=1}^{k}\lambda_j$$

- Not so far off Ashbaugh-Benguria (In 2D, $\lambda_2/\lambda_2 \leq 3$ rather than 2.5387….

Universal spectral patterns

- Numerous extensions by same method: Cook up a trial function for min-max by multiplying $\{u_1, \ldots\ u_k\}$ by a coordinate function x_α. For example:

$$1 \leq \frac{4}{d}\frac{1}{k}\sum_{j=1}^{k}\frac{\lambda_j}{\lambda_{k+1}-\lambda_j}$$

Compare to PPW: $\lambda_{k+1} - \lambda_k \leq \frac{4}{d}\,\frac{1}{k}\sum_{j=1}^{k}\lambda_j$

Hile-Protter 1980

Universal spectral patterns

- Numerous extensions by same method: Cook up a trial function for min-max by multiplying $\{u_1, \dots\ u_k\}$ by a coordinate function x_α.

- Before 1993, these were all lousy for large k, as we knew by Weyl.

H.C. Yang, unpublished preprint, 1991,3,5

$$\sum_{j=1}^{k} (\lambda_{k+1} - \lambda_j)^2 \leq \frac{4}{d} \sum_{j=1}^{k} \lambda_j (\lambda_{k+1} - \lambda_j)$$

H.C. Yang, unpublished preprint, 1991,3,5

$$\sum_{j=1}^{k} (\lambda_{k+1} - \lambda_j)^2 \leq \frac{4}{d} \sum_{j=1}^{k} \lambda_j (\lambda_{k+1} - \lambda_j)$$

Some people will do anything for an eigenvalue!

H.C. Yang, unpublished preprint, 1991,3,5

$$\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j)^2 \le \frac{4}{d}\sum_{j=1}^{k}\lambda_j(\lambda_{k+1}-\lambda_j)$$

If you estimate with the Weyl law $\lambda_j(\Omega) \sim j^{2/d}$, the two sides agree with the correct constant.

What is the underlying idea behind these universal spectral relations and how can you extract consequences you can understand?

Commutators of operators

- $[H, G] := HG - GH$
- $[H, G]\, u_k = (H - \lambda_k)\, G\, u_k$
- If $H=H^*$,

$$\langle u_j,[H, G]\, u_k\rangle = (\lambda_j - \lambda_k)\, \langle u_j,Gu_k\rangle$$

Commutators of operators

- $[G, [H, G]] = 2\,GHG - G^2H - HG^2$
- Etc., etc. Try this one:

$$\langle u_j, [G, [H, G]]\, u_j \rangle = \sum_{k:\lambda_k \neq \lambda_j} (\lambda_k - \lambda_j) |G_{jk}|^2$$

Commutators of operators

- For (flat) Schrödinger operators, $G=x_\alpha$,
  - $[H, G] = - 2\, \partial/\partial x_\alpha$
  - $[G, [H, G]] = 2$

Commutators of operators

$$1 = \frac{4}{d} \sum_{k:\lambda_k \neq \lambda_j} \frac{|\langle u_j, \nabla u_k \rangle|^2}{\lambda_k - \lambda_j}$$

Commutators of operators

$$1 = \frac{4}{d} \sum_{k:\lambda_k \neq \lambda_j} \frac{|\langle u_j, \nabla u_k \rangle|^2}{\lambda_k - \lambda_j}$$

Compares with Hile-Protter:

$$1 \leq \frac{4}{d}\frac{1}{k} \sum_{j=1}^{k} \frac{\lambda_j}{\lambda_{k+1} - \lambda_j}$$

Commutators of operators

$$1 = \frac{4}{d} \sum_{k:\lambda_k \neq \lambda_j} \frac{|\langle u_j, \nabla u_k \rangle|^2}{\lambda_k - \lambda_j}$$

Variant: If H is the Laplace-Beltrami operator + V(x), there is an additional term involving mean curvature.

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- The counting function,

  $N(z) := \#(\lambda_k \leq z)$

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- The counting function,

  $N(z) := \#(\lambda_k \leq z)$
- Integrals of the counting function, known as *Riesz means* (Safarov, Laptev, etc.):

$$R_\rho(z) := \sum_j (z - \lambda_j)_+^\rho$$

- Chandrasekharan and Minakshisundaram, 1952

Trace identities imply differential inequalities

$$\left(1+\frac{4}{d}\right)R_2(z)-\frac{4z}{d}R_1(z)\leq 0.$$

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$$\left(1+\frac{4}{d}\right)R_2(z)-\frac{4z}{d}R_1(z)\leq 0.$$

$$\left(1+\frac{4}{d}\right)R_2(z)-\frac{2z}{d}R_2'(z)\leq 0.$$

Trace identities imply differential inequalities

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$$\left(1+\frac{4}{d}\right)R_2(z)-\frac{2z}{d}R_2'(z)\leq 0.$$

It follows that $R_2(z)/z^{2+d/2}$ is an increasing function.

Understandable eigenvalue bounds from inequalities on $R_\rho(z)$:

- Legendre transform – gives the ratio bounds
- Laplace transform – gives a lower bound on Z(t) = Tr(exp(-Ht)).
- The latter implies a lower bound on the spectral zeta function (through the Mellin transform)

Summary

- Eigenvalues fall into universal patterns, with distinct statistical signatures
- Spectra contolled by “kinetic energy” and geometry
- Underlying idea: Identities for commutators
- Extract information about λ_k by differential inequalities, transforms.