Means of chords and related problems of geometric optimization in the plane

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Abstract

- Abstract: I'll discuss some problems of geometric optimization that began with an attempt to understand the means of chords on a closed planar curve of fixed length. By a chord we refer to the length of the line segment joining two points on the curve, differing by arclength \alpha. One can consider different means of this quantity, for example L^p means with respect to arclength, or with respect to a weight proportional to curvature. For example, for $1 \leq p \leq 2$, in the unweighted case the means of chords are shown to be maximized by the circle. The situation is different for sufficiently high p and for the weighted means we consider, and we can identify some cases of optimum while the situation is open in other cases. In the weighted case we can impose the constraint of convexity and identify a wider family of convex functionals for which the maximizing shapes are triangles or segments. Among other things, the more general problems include maximizing Hausdorff distances between sets, under some geometric constraints.

Some physical motivation: An electron near a charged thread

LMP 2006, with Exner and Loss

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)$$

Fix the length of the thread. What shape binds the electron the least tightly? Conjectured for some years that answer is circle.

Reduction to an isoperimetric problem of classical type.

Is it true that:

$$\int_0^L |\Gamma(s+u) - \Gamma(s)| \, \mathrm{d}s \leq \frac{L^2}{\pi} \sin \frac{\pi u}{L}$$

#2: An electromagnetic problem of classical type.

If a uniformly charged thread (deformable closed loop) is put into a tub of gelatin, what shape will it assume?

#2: An electromagnetic problem of classical type.

Minimize the expression:

$$\int_{C\times C} \left|\Gamma(s) - \Gamma(s')\right|^{-2} ds ds',$$

which after a change of variable requires minimizing the integral over u of

$$\int_{C} \left|\Gamma(s) - \Gamma(s+u)\right|^{-2} ds.$$

A family of isoperimetric conjectures for p > 0:

$$C_L^p(u): \qquad \int_0^L |\Gamma(s+u)-\Gamma(s)|^p \, \mathrm{d}s \le \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L},$$

$$C_L^{-p}(u): \qquad \int_0^L |\Gamma(s+u)-\Gamma(s)|^{-p} \, \mathrm{d}s \ge \frac{\pi^p L^{1-p}}{\sin^p \frac{\pi u}{L}},$$

Right side corresponds to circle.

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The case C^{-1} arises in an electromagnetic problem: minimize the electrostatic energy of a charged nonconducting thread.

Proposition. 2.1.

$C_L^p(u)$ *implies* $C_L^{p'}(u)$ *if* $p > p' > 0.$

$C_L^p(u)$ *implies* $C_L^{-p}(u)$

First part follows from convexity of x → x^a for a > 1:

$$\frac{L^{1+p}}{\pi^p}\sin^p\frac{\pi u}{L} \;\geq\; \int_0^L \left(|\Gamma(s+u)-\Gamma(s)|^{p'}\right)^{p/p'} \mathrm{d}s$$
$$\geq\; L\left(\frac{1}{L}\int_0^L |\Gamma(s+u)-\Gamma(s)|^{p'}\,\mathrm{d}s\right)^{p/p'} .$$

Proof when p = 2

$$\Gamma(s) = \sum_{0 \neq n \in \mathbb{Z}} c_n \, \mathrm{e}^{ins}$$

$$c_{-n} = \bar{c}_n \, .$$

$$\dot{\Gamma}(s) = i \sum_{0 \neq n \in \mathbb{Z}} n c_n \, \mathrm{e}^{ins} \, .$$

By assumption, $|\dot{\Gamma}(s)| = 1$, and hence from the relation

$$2\pi = \int_0^{2\pi} |\dot{\Gamma}(s)|^2 \, \mathrm{d}s = \int_0^{2\pi} \sum_{0 \neq m \in \mathbb{Z}} \sum_{0 \neq n \in \mathbb{Z}} nm \, c_m^* \cdot c_n \, \mathrm{e}^{i(n-m)s} \, \mathrm{d}s \,,$$

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 = 1 \,. \tag{2.5}$$

At first this appears to greatly weaken the condition that $\dot{\Gamma}$ is a unit vector for each s. However, since the case of equality in

$$(2\pi)^2 = \left(\int \left| \dot{\Gamma} \right| ds \right)^2 \leq 2\pi \int \left| \dot{\Gamma} \right|^2 ds$$

requires $\dot{\Gamma} = cst.a.e.$, in fact it is fully equivalent.

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$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 = 1 \,. \tag{2.5}$$

$$\int_0^{2\pi} \left| \sum_{0 \neq n \in \mathbb{Z}} c_n \left(\mathrm{e}^{inu} - 1 \right) \mathrm{e}^{ins} \right|^2 \mathrm{d}s = 8\pi \sum_{0 \neq n \in \mathbb{Z}} |c_n|^2 \left(\sin \frac{nu}{2} \right)^2 \,,$$

Inequality equivalent to

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 \left(\frac{\sin \frac{nu}{2}}{n \sin \frac{u}{2}} \right)^2 \leq 1.$$

It is therefore sufficient to prove that

$$|\sin nx| \leq n \sin x$$

Inductive argument based on

$$(n+1)\sin x \mp \sin(n+1)x = n\sin x \mp \sin nx \cos x + \sin x(1 \mp \cos nx)$$

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The conjecture is false for $p = \infty$. The family of maximizing curves for $||\Gamma(s+u) - \Gamma(s)||_\infty$ consists of all curves that contain a line segment of length > s.

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At what critical value of p does the circle stop being the maximizer?

This problem is open. We calculated $||\Gamma(s+u) - \Gamma(s)||_p$ for some examples:

Two straight line segments of length π:

$||\Gamma(s+u) - \Gamma(s)||_p^p = 2^{p+2}(\pi/2)^{p+1}/(p+1)$.

Better than the circle for $p > 3.15296\ldots$

What about p > 2?

Examples that are more like the circle are not better than the circle until higher p:

Stadium, small straight segments $p > 4.27898\ldots$

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Polygon with many sides, $p > 6$

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Stadium, small straight segments   p > 4.27898…

Polygon with many sides, p > 6

Polygon with rounded edges, similar.

Circle is local maximizer for $p < p_c$

Theorem 2 *For a fixed arc length $u \in (0, \frac{1}{2}L]$ define*

$$p_c(u) := \frac{4 - \cos\left(\frac{2\pi u}{L}\right)}{1 - \cos\left(\frac{2\pi u}{L}\right)}, \tag{6}$$

then we have the following alternative. For $p > p_c(u)$ the circle is either a saddle point or a local minimum, while for $p < p_c(u)$ it is a local maximum of the map $\Gamma \mapsto c^p_\Gamma(u)$.

Reduction to an isoperimetric problem of classical type.

$$\int_0^L |\Gamma(s+u) - \Gamma(s)|\,\mathrm{d}s \le \frac{L^2}{\pi}\sin\frac{\pi u}{L}$$

Science is full of amazing coincidences!

Mohammad Ghomi and collaborators had considered and proved similar inequalities in a study of knot energies, A. Abrams, J. Cantarella, J. Fu, M. Ghomi, and R. Howard, *Topology*, 42 (2003) 381-394! They relied on a study of mean lengths of chords by G. Lükö, Isr. J. Math., 1966.

Some problems of optimization with convexity constraints.

Suppose we weight the means of chords proportionally to curvature.

Some problems of optimization with convexity constraints.

Suppose we weight the means of chords proportionally to curvature. *This is equivalent to the uniform measure on the set of normal vectors (the one-dimensional Gauss sphere)*, and it invites constraint of convexity.

Some problems of optimization with convexity constraints.

It is not difficult to see that the circle is now the *minimizer*, among convex curves of fixed perimeter, all offsets $0 < u \leq \pi$. What about the maximizer?

The support function

h(θ) := distance to support plane normal to $\mathbf{n}(\theta)$

$$\mathbf{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \mathbf{t} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

The support function

Let K be a plane convex set.
The support function h_K of K is defined by:

$$h_K(\theta) := \max\{x \cdot e^{i\theta} : x \in K\}\,.$$

The perimeter $P(K)$ of the convex set is given by:

$$P(K) = \int_0^{2\pi} h_K(\theta)\, d\theta\,.$$

The Steiner point $s(K)$ of the convex set is defined by:

$$s(K) = \frac{1}{\pi} \int_0^{2\pi} h_K(\theta) e^{i\theta}\, d\theta\,.$$

The support function

The support function gives an easy characterization of convex sets:

$$K \text{ is a convex set} \iff h_K'' + h_K \text{ is a positive measure}$$

The polygons are also well characterized

$$K \text{ is a polygon} \iff h_K'' + h_K = \sum_{j=1}^{n} a_j \delta_{\theta_j}$$

where $a_1, a_2, \ldots, a_n$ and $\theta_1, \theta_2, \ldots, \theta_n$ denote the lengths of the sides and the angles of the corresponding outer normals.

Geometric quantities are often easy to express in terms of the support function.

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$$\mathbf{r} = h(\theta)\mathbf{n} + h'(\theta)\mathbf{t}$$

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Chords and their Fourier series

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Chords and their Fourier series

$$x + iy = (h(\theta) + h'(\theta))e^{i\theta} = e^{i\theta} \sum h_k (1-k) e^{ik\theta}$$

$$\|\mathbf{r}(\theta+\alpha) - \mathbf{r}(\theta)\|_{L^2}^2 = \int_0^{2\pi} |\sum h_k (1-k)(e^{i(k+1)\alpha} - 1) e^{ik\theta}|^2 d\theta$$

Geometric quantities are often easy to express in terms of the support function.

Chords and their Fourier series. Finally:

$$\|\mathbf{r}(\theta+\alpha)-\mathbf{r}(\theta)\|_{L^2}^2 = 2\pi \sum |h_k|^2 (1-k)^2 2^2 \sin^2\left(\frac{(k+1)\alpha}{2}\right)$$

Some problems of optimization with convexity constraints.

We are asking to maximize a constrained convex functional, so we expect the optimizer to be extremal in an appropriate sense.

Some problems of optimization with convexity constraints.

Definition:

K is *indecomposable* (in M) if
$K = (1 - t)K_0 + tK_1$ (with $K_0, K_1 \in M$)
implies that K_0, K_1 are homothetic to K.

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Theorem In $\mathbb{R}^2$, the indecomposable convex sets are the triangles and the segments.

Corollary Any maximizer of a strictly convex functional in the plane is a segment or a triangle.

Example: The farthest convex set

We consider the class

$$\mathcal{A} = \{K \subset \mathbb{R}^2; K \text{ convex set }, P(K) = 2\pi, s(K) = O\}$$

where $P(K)$ denotes the perimeter of K and $s(K)$ its Steiner point.

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More precisely, we want to answer the following
Question 1: Let C be given in $\mathcal{A}$, what is the farthest convex set K_C such that $d(K_C, C) = \max_{K\in\mathcal{A}} d(K, C)$.

Geometric observations

Classical fact: If we maximize a strictly convex function over a convex domain $\mathcal{A}$, the maximum is attained at extreme points of $\mathcal{A}$.

For domains, convexity or concavity properties are known as Brunn-Minkowski inequalities. For example, in the plane, $|K|^{1/2}$ or $\lambda_1(\Omega)^{-1/2}$ are strictly concave:

$$|(1-t)K_0 + tK_1|^{1/2} \geq (1-t)|K_0|^{1/2} + t|K_1|^{1/2}$$

with equality iff K_0, K_1 are homothetic.

The support function and metrics on the set of sets

The Hausdorff distance can be defined using the support functions:

$$d_H(K,L) = \|h_K - h_L\|_\infty.$$

We can also define a L^2 distance (Mc Clure and Vitale) by

$$d_2(K,L) := \left(\int_0^{2\pi} |h_K - h_L|^2 \, d\theta\right)^{1/2}.$$

The farthest convex set

Theorem Let J be a functional defined by

$$J(K) := \int_0^{2\pi} a\,h_K^2 + b\,{h'_K}^2 + c\,h_K + d\,h'_K\,d\theta$$

(example the L^2 distance: $J(K) = \int_0^{2\pi} (h_K - h_C)^2\,d\theta$). Then every local maximizer of the functional J within the class $\mathcal{A}$ is either a segment or a triangle.

Proof: Use indecomposability!

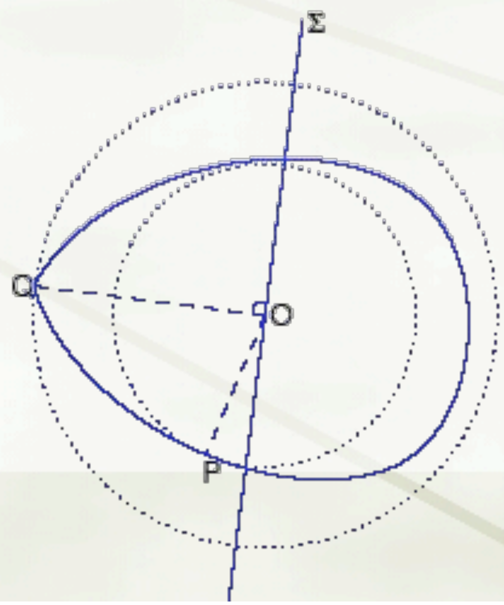
Corollary The farthest convex set for the L^2 distance is either a segment or a triangle.

The farthest convex set

Theorem [farthest convex set for Hausdorff distance]
If C is a given convex set in the class $\mathcal{A}$, then the convex set K_C for which

$$d_H(C, K_C) = \max\{d_H(C, K) : K \in \mathcal{A}\}$$

is a segment.



Lemma: The perimeter inequalities

Theorem Let K be any plane convex set with its Steiner point at the origin. Then

$$\max h_K \leq \frac{P(K)}{4} \leq \min h_K + \max h_K,$$

where both inequalities are sharp and saturated by any line segment.

The first inequality is due to P. Mc Mullen. It implies that the diameter of $\mathcal{A}$ is less than $\pi/2$.

Three points lemma

Analytic approach

We recall that K is convex iff $h''_K + h_K$ is a positive measure. We want to perform variations preserving convexity.

Lemma[T. Lachand-Robert,M. Peletier, J. Lamboley, A. Novruzi] If suppt$(h'' + h)$ has at least 3 points in $(0, \varepsilon)$, there exists v compactly supported in $(0, \varepsilon)$ such that $h + tv$ is the support function of a convex set.

Consequence: If $J(K) = j(h_K)$ is strictly "locally concave" in h, the minimizers have to be polygons.

The End