Have you heard what's going round?

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Research Horizons
Georgia Tech
22 April 2009

Past graduate students and their dissertations in pure math, applied math, or both

- Edward L. Green, *Spectral Theory of Laplace-Beltrami Operators with Periodic Metric*
- Glenn E. James, *Models of Intracavity Frequency Doubled Lasers*
- Patricia L. Michel, *Eigenvalue Gaps for Self-Adjoint Operators*
- W. Richard O'Connell, *Estimates for the St. Petersburg Game*
- Roman Svirsky, *Potentials Producing Maximally Sharp Resonances*
- Dale T. Smith, *Exponential Decay of Resolvents of Banded Infinite Matrices and Asymptotics of Solutions of Linear Difference Equations*
- Xue-Feng Yang, *Nodal Sets in both Ordinary and Partial Differential Equations (GT thesis award 1998)*
- Secondary advisor: Qian Chen, *A physical short-channel threshold voltage model for undoped symmetric double-gate MOSFETs* (ECE)
- Edward C. White, Jr., *Polar and Legendre Duality in Convex Geometry and Geometric Flows* (MS thesis)

Where did the doctoral students go?

- Edward L. Green, North Georgia College & State U.
- Glenn E. James, Dean of Math, Sci., Eng. at Univ. of the Incarnate Word (Air Force Academy)
- Patricia L. Michel, Pasadena City College
- W. Richard O'Connell, Crédit Suisse (Greenwich Partners)
- Roman Svirsky, GEICO (Tulane, U. Tenn. Knoxville)
- Dale T. Smith, VICIS Capital (N. Texas, Wake Forest)
- Xue-Feng Yang, Director, Institute of Applied Mathematics and Software Engineering, Zhejiang Gongshang University (McMaster U.)

You might be a mathematician if…

You might be a mathematician if … this appeals to you

§ 32. Eleven Properties of the Sphere

We have already become acquainted with the surfaces of vanishing Gaussian curvature. We shall now look for the surfaces of constant positive or negative curvature. By far the simplest and most important surface of this type is the sphere. A thorough study of the sphere would in itself provide sufficient material for a whole book. We shall here present only eleven properties that have a particularly strong appeal to the visual intuition. We shall at the same time become acquainted with several properties that are of importance not only for the geometry of the sphere but also for the general theory of surfaces. With regard to each property to be described we shall inquire whether it defines the sphere uniquely or whether there are other surfaces having the given property.

1. *The points of a sphere are equidistant from a fixed point. Also, the ratio of the distances of its points from two fixed points is constant.*

The first of these two properties constitutes the elementary definition of the sphere and consequently defines the sphere uniquely. The fact that the sphere has the second property as well, can be ascertained very easily by analytical methods. On the other hand, the second property defines not only the sphere but the plane as well. For, a plane is obtained if, and only if, the constant ratio is equal to unity. The plane obtained in this case is the plane of symmetry of the two fixed points.

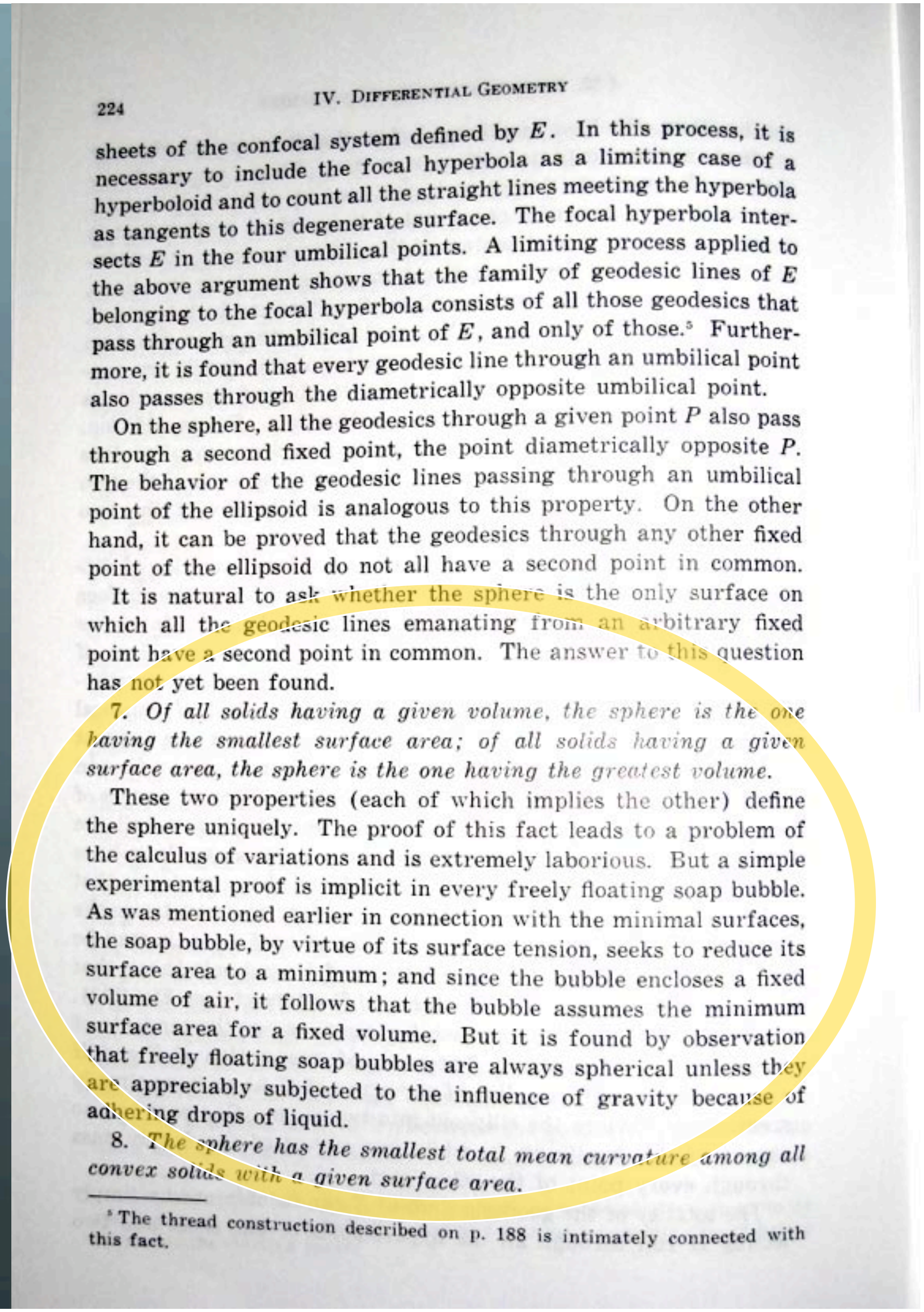
2. *The contours and the plane sections of the sphere are circles.*

In the discussion of the second-order surfaces we mentioned the theorem that all the plane sections and contours of such surfaces are conics. In the case of a sphere, all these conics are circles. This property defines the sphere uniquely. From the observation that the shadow of the earth at a lunar eclipse is always a circle we may therefore infer that the earth is spherical.

3. *The sphere has constant width and constant girth.*

The term *constant width* denotes the property, of a solid, that the distance between any pair of parallel tangent planes is constant. Thus a sphere can be rolled arbitrarily between two parallel tangent planes. It would seem plausible that the sphere is uniquely defined by this property. In actual fact, however, there are numerous other closed convex surfaces, some of them without any singularities, whose

You might be a mathematician if … this appeals to you



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sheets of the confocal system defined by E. In this process, it is necessary to include the focal hyperbola as a limiting case of a hyperboloid and to count all the straight lines meeting the hyperbola as tangents to this degenerate surface. The focal hyperbola intersects E in the four umbilical points. A limiting process applied to the above argument shows that the family of geodesic lines of E belonging to the focal hyperbola consists of all those geodesics that pass through an umbilical point of E, and only of those.[5] Furthermore, it is found that every geodesic line through an umbilical point also passes through the diametrically opposite umbilical point.

On the sphere, all the geodesics through a given point P also pass through a second fixed point, the point diametrically opposite P. The behavior of the geodesic lines passing through an umbilical point of the ellipsoid is analogous to this property. On the other hand, it can be proved that the geodesics through any other fixed point of the ellipsoid do not all have a second point in common.

It is natural to ask whether the sphere is the only surface on which all the geodesic lines emanating from an arbitrary fixed point have a second point in common. The answer to this question has not yet been found.

7. *Of all solids having a given volume, the sphere is the one having the smallest surface area; of all solids having a given surface area, the sphere is the one having the greatest volume.*

These two properties (each of which implies the other) define the sphere uniquely. The proof of this fact leads to a problem of the calculus of variations and is extremely laborious. But a simple experimental proof is implicit in every freely floating soap bubble. As was mentioned earlier in connection with the minimal surfaces, the soap bubble, by virtue of its surface tension, seeks to reduce its surface area to a minimum; and since the bubble encloses a fixed volume of air, it follows that the bubble assumes the minimum surface area for a fixed volume. But it is found by observation that freely floating soap bubbles are always spherical unless they are appreciably subjected to the influence of gravity because of adhering drops of liquid.

8. *The sphere has the smallest total mean curvature among all convex solids with a given surface area.*

[5] The thread construction described on p. 188 is intimately connected with this fact.

814 BC, the original isoperimetric theorem

Dido Purchases Land for the Foundation of Carthage. Engraving by Matthäus Merian the Elder, in *Historische Chronica*, Frankfurt a.M., 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

How Dido might have proved the isoperimetric theorem

- Might as well suppose region is convex.

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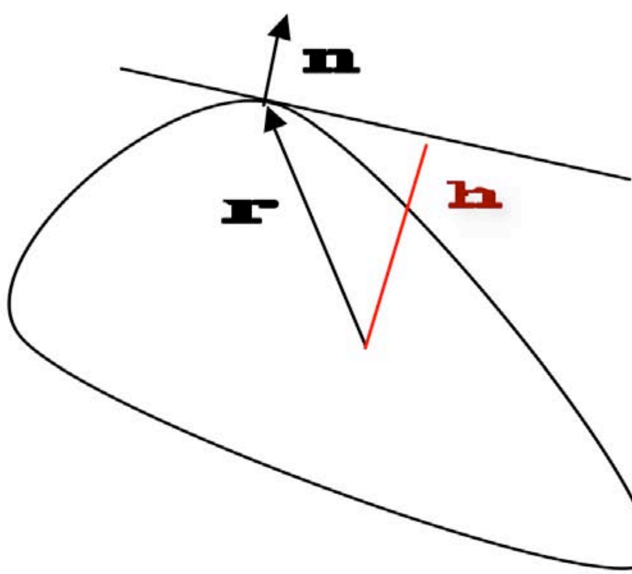
- Might as well suppose region is convex.
- A convex region can be described in terms of its *gauge function* or its *support function*.
  - $h(\theta)$ = distance to support plane/line at angle θ.

How best to describe a convex body?

r - boundary point.

Support function $h(\omega)$ for D - ∂D is a continuous image of S^{d-1} .
I.e., h to be regarded as a function of the unit normal vectors ω.

$h(\theta) = \mathbf{r} \bullet \mathbf{n}$



How Dido might have proved the isoperimetric theorem

- Might as well suppose region is convex.
- A convex region can be described in terms of its *gauge function* or its *support function.*
  - $h(\theta)$ = distance to support line at normal angle θ.
  - perimeter is integral of h
  - area is integral of $h^2 + h\, h_{\theta\theta}$.

How Dido might have proved the isoperimetric theorem

- Might as well suppose region is convex.
- A convex region can be described in terms of its *gauge function* or its *support function.*

$$\mathbf{p} = \int_0^{2\pi} \mathbf{h}\, \mathbf{d}\theta, \qquad \mathbf{A} = \frac{1}{2}\int_0^{2\pi} \left(\mathbf{h}^2 + \mathbf{h}\mathbf{h}_{\theta\theta}\right) \mathbf{d}\theta = \frac{1}{2}\int_0^{2\pi} \left(\mathbf{h}^2 - \mathbf{h}_\theta^2\right) \mathbf{d}\theta$$

- **Use Fourier series.**

$$\mathbf{h} = \sum_{\mathbf{k}} \mathbf{h}_{\mathbf{k}} \exp(\mathbf{i}\mathbf{k}\theta), \qquad \mathbf{h}_{\mathbf{k}} := \frac{1}{2\pi}\int_0^{2\pi} \mathbf{h} \exp(-\mathbf{i}\mathbf{k}\theta)\, \mathbf{d}\theta$$

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- Use Fourier series. (Too bad Euler wouldn't discover them until 2594 years later!)

How Dido might have proved the isoperimetric theorem

$$\mathbf{p} = \mathbf{h_0}, \qquad \mathbf{A} = \frac{1}{2}\int_0^{2\pi} \left(\mathbf{h^2} + \mathbf{h}\mathbf{h}_{\theta\theta}\right) \mathbf{d}\theta = \frac{1}{2}\int_0^{2\pi} \left(\mathbf{h^2} - \mathbf{h}_\theta^2\right) \mathbf{d}\theta$$

$$\mathbf{h} = \sum_{\mathbf{k}} \mathbf{h_k} \exp(\mathbf{ik}\theta), \qquad \mathbf{h_k} := \frac{1}{2\pi}\int_0^{2\pi} \mathbf{h} \exp(-\mathbf{ik}\theta)\, \mathbf{d}\theta$$

$$\mathbf{A} = \frac{1}{4\pi}\sum_{\mathbf{k}} \left(|\mathbf{h_k}|^2 - \mathbf{k}^2|\mathbf{h_k}|^2\right) = \frac{\mathbf{p}^2}{4\pi} - \text{stuff}$$

$\blacksquare$

Spectral geometry, or *What do eigenvalues tell us about shapes?*

- M. Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly*, 1966.

Normal modes of vibration:

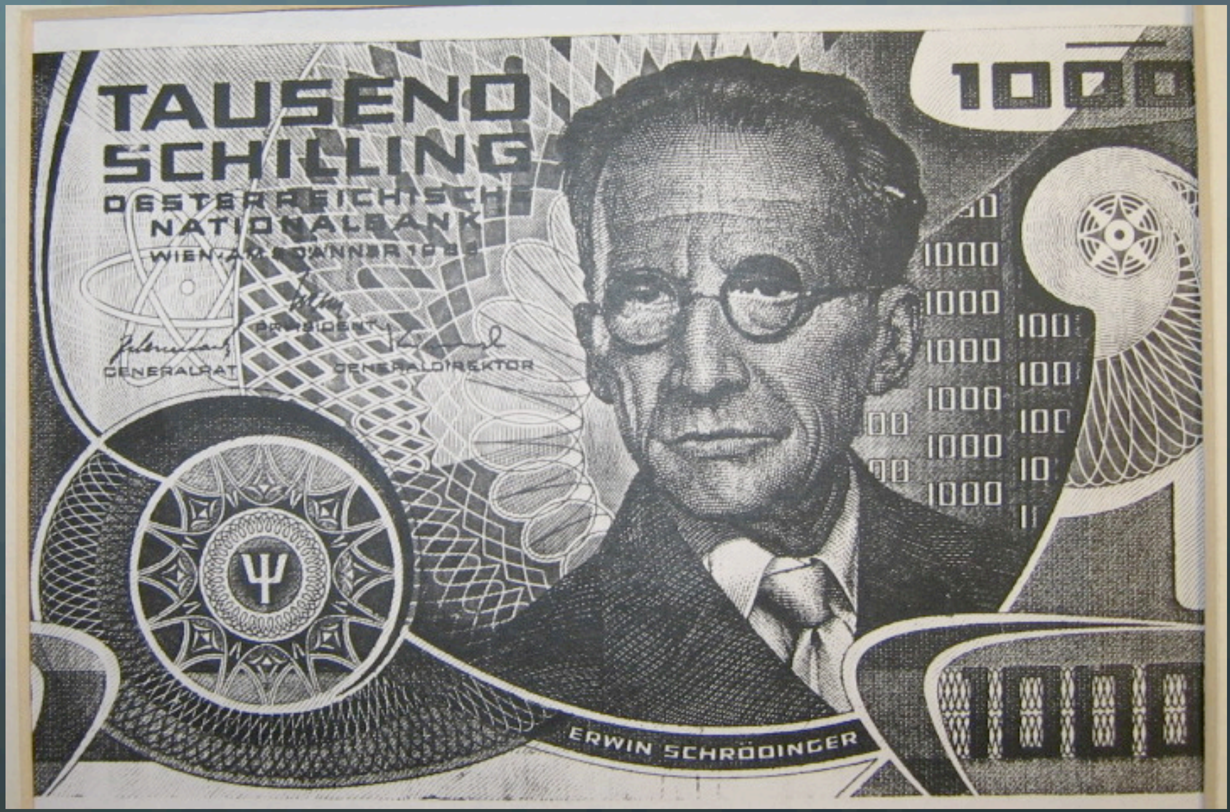
$$-\Delta u = (\omega/c)^2 u =: \lambda u.$$

Spectral geometry, or *What do eigenvalues tell us about shapes?*

- M. Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly*, 1966.
- Already in 1946, G. Borg considered whether you could hear the density of a guitar string, *but he failed to think of such a colorful title.*

Schrödinger operators

$$-\Delta + V(x)$$



Inverse spectral theory

- Asking two questions: If we look for eigenvalues (normal modes) of the differential operator

  $$- \Delta + V(x)$$

  acting on functions on a region (or manifold or surface) M,

  What do we know about V(x) *or* M?

Can you hear the shape of a drum, or the density of a string, or the strength of an interaction?

- Can you determine the domain Ω &/or the potential

  $V(x)$ from the eigenvalues of the Laplace or Schrödinger operator?

So, can you hear the density of a string?

- I.e., find $V(x)$ given the eigenvalues of

  $-d^2/dx^2 + V(x)$

  on an interval with some reasonable boundary conditions (Dirichlet, Neumann, periodic)?

Can you even hear the density of a string?

- No!   In 1946 Borg showed there is usually an infinite-dimensional set of "isospectral" V(x).

Well, can you hear the shape of a drum?

So, can you hear the shape of a drum?

Gordon, Webb, and Wolpert, 1991

Can you hear the interaction in quantum mechanics from scattering experiments?

- **No! Bargmann exhibited two different potentials with the same scattering data in 1949.**

Can you hear the interaction in quantum mechanics from scattering experiments?

- **No! Bargmann exhibited two different potentials with the same scattering data in 1949, *thereby destroying the careers of whole tribes of chemists and causing bad blood between the disciplines ever since!***

Some things are "audible"

- You can hear the area of the drum, by the *Weyl asymptotics*:
- For the drum problem

$$\lambda_k \sim C_d\ (\mathrm{Vol}(\Omega)/k)^{2/d}.$$

(A mathematician's drum can be d-dimensional, and even be a curved manifold.)

Some things are "audible"

- The Schrödinger equation also exhibits Weyl asymptotics, which determine both
  - the volume of the region, and
  - the average of V(x).

To extremists, things tend to look simple…

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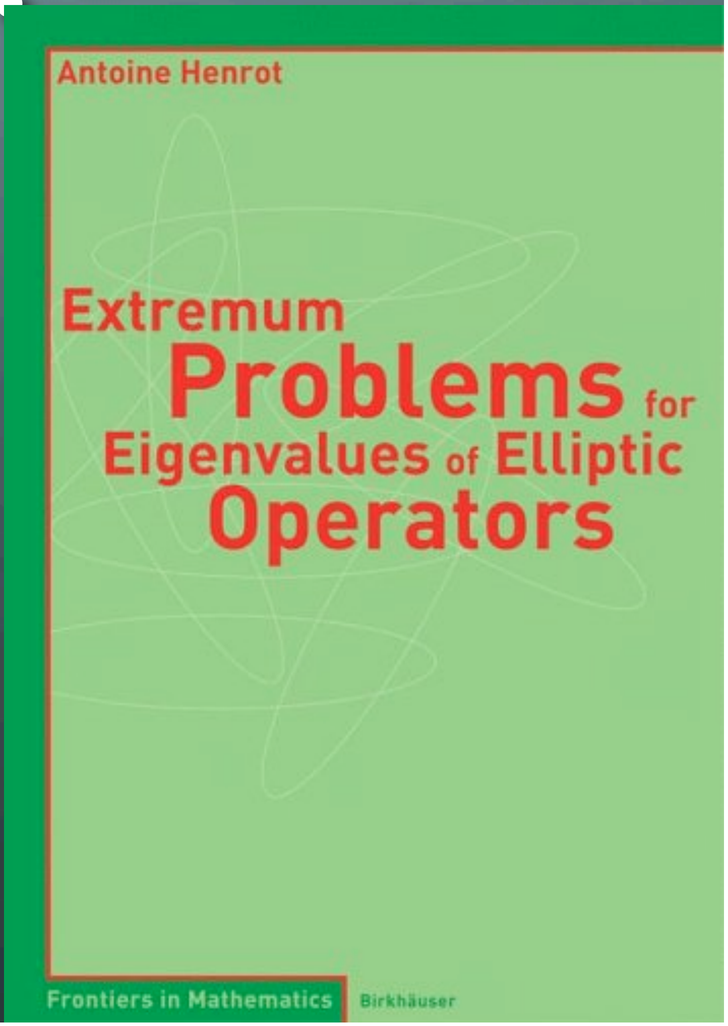

Classic extreme spectral theorem

- Rayleigh conjectured, and Faber and Krahn proved, that if you fix the area of a drum, the lowest eigenvalue is minimized uniquely by the disk. This requires Dirichlet boundary conditions - the displacement is 0 at the edge.

Classic extreme spectral theorem

- Rayleigh conjectured, and Faber and Krahn proved, that if you fix the area of a drum, the lowest Dirichlet eigenvalue is minimized uniquely by the disk.
- Seemingly, rounder $\Rightarrow$ deeper tone.
- True in any dimension, many proofs.

Classic extreme spectral theorem

- Rayleigh conjectured, and Faber and Krahn proved, that if you fix the area of a drum, the lowest eigenvalue is minimized uniquely by the disk.
- Seemingly, rounder $\Rightarrow$ deeper tone
- However, if your drum must be annular (fixing edge lengths and width), circular geometry *maximizes* λ_1.

Some extreme shapes

- What shape minimizes λ_2?

Some extreme shapes

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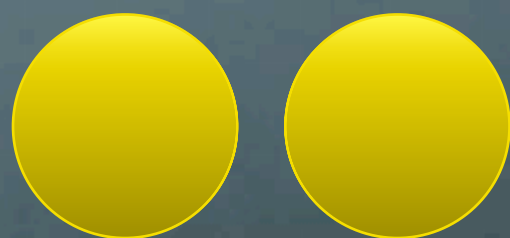
  - Answer (Polya-Szegö):



Some extreme shapes

- What shape maximizes λ_2/λ_1?

Some extreme shapes

- What shape maximizes λ_2/λ_1?
  - Answer (Ashbaugh-Benguria):

Some extreme shapes

- What if you fix the volume and minimize the third Dirichlet eigenvalue (2D)?

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  - Conjectured answer:

Some extreme shapes

- What if you fix the volume and minimize the fourth Dirichlet eigenvalue (2D)?

Some extreme shapes

- What if you fix the volume and minimize the fourth Dirichlet eigenvalue?
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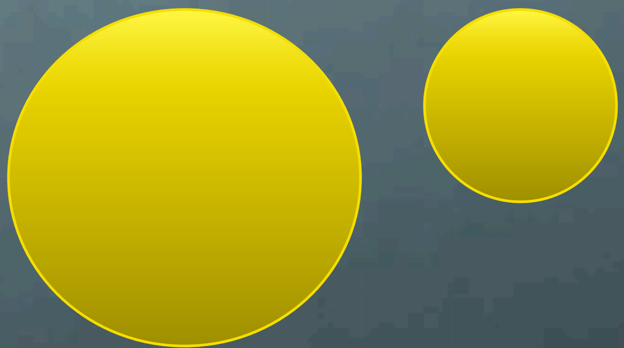


Some extreme shapes
(closed manifolds this time)

- **If there is no boundary, the lowest eigenvalue of the Laplacian is trivial, $-\Delta\, 1 = 0 * 1$. What if you fix the volume and *max*imize the first nontrivial eigenvalue?**

Some extreme shapes

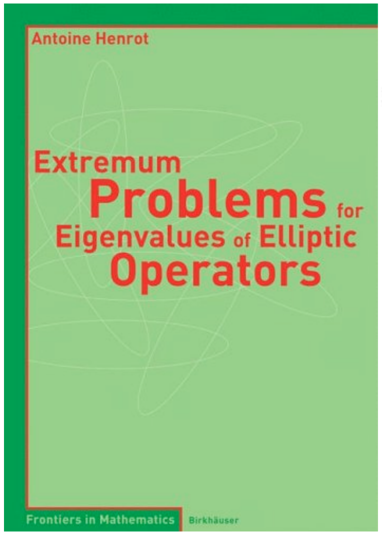
(closed manifolds this time)

- If there is no boundary, the lowest eigenvalue of the Laplacian is trivial, $-\Delta 1 = 0 * 1$. What if you fix the volume and maximize the first nontrivial eigenvalue?
  - Answer (Hersch):

Is the extremum always a union of round shapes?

Is the extremum always a union of round shapes?

5.4. *Case of higher eigenvalues* 83



No	Optimal union of discs		Computed shapes	
3		46.125		46.125
4		64.293		64.293
5		82.462		**78.47**
6		92.250		**88.96**
7		110.42		**107.47**
8		127.88		**119.9**
9		138.37		**133.52**
10		154.62		**143.45**

`You may seek it with trial functions---and seek it with care;
You may hunt it with rearrangements and hope;
You may perturb the boundary with a lump here and there;
You may fool it with some series rope-a-dope---'

Apologies to Lewis Carroll.

`You may seek it with trial functions---and seek it with care;

The Rayleigh-Ritz inequality

$$\inf sp(A) = \inf_{\|\varphi\|=1} \langle \varphi, A\varphi \rangle =$$

`You may seek it with trial functions---and seek it with care;

The min-max principle

λ_k = max min $<\varphi, A\,\varphi>$,

φ normalized and orthogonal

φ to a k-1 dimensional subspace.

Schrödinger with Rayleigh and Ritz

For $H = -\nabla^2 + V(x)$, φ smooth,

$$\inf sp(H) \leq \frac{\int |\nabla\varphi|^2 + V(x)|\varphi|^2}{\int |\varphi|^2}$$

"Rayleigh quotient."

Potential energy coming from curvature

- Thin structures– e.g. quantum wires and waveguides.

Nanoelectronics

- Quantum wires
- Semi- and non-conducting “threads”
- Quantum waveguides

In simple but reasonable mathematical models, the Schrödinger equation responds to the geometry of the structure either through the boundary conditions or through an “effective potential.”

Thin-domain Schrödinger operator

$$-\nabla_{\|}^2 + q(\mathbf{x}) = -\Delta_\Omega + q(\mathbf{x})$$

$$q(\mathbf{x}) = \frac{1}{4}\left(\sum_{j=1}^{d} \kappa_j\right)^2 - \frac{1}{2}\sum_{j=1}^{d} \kappa_j^2$$

Spherical Shells. Infinitesimal case.

Suppose $\Omega \cong S^2 \subset \mathbb{R}^3$. $\lambda_{1,2}(-\Delta + q(\kappa))$?

$$q(\kappa) = -\frac{(\kappa_1 - \kappa_2)^2}{4}$$ (similar for $-c^{\kappa_1 \kappa_2}$, etc.)

$$\lambda_1 \leq \frac{\int |\nabla 1|^2 + \int q(\kappa)}{\int 1} = 0 + \langle q \rangle \leq 0$$

= iff sphere. (WHY?)

Yet another "isoperimetric theorem," this time for λ_2.

- Consider the thin-domain operator on a closed, simply connected surface in R^3,

  $- \nabla^2 - (\kappa_2 - \kappa_1)^2/4.$

- The ground state is maximized (at 0) by the sphere. Let's fix the area and ask after the maximum of the second eigenvalue.

Yet another "isoperimetric theorem," this time for λ_2.

Eigenfunctions of a self-adjoint operator, with different eigenvalues, are orthogonal. Therefore if we search over φ orthogonally to u_1,

$$\lambda_2 \leq \langle\varphi, A\,\varphi\rangle/\|\varphi\|^2.$$

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Problem: We don't know u_1 *a priori*. One way around this is a lemma of J. Hersch:

Yet another "isoperimetric theorem," this time for λ_2.

Lemma. *(J. Hersch). Let Ω be a two-dimensional, closed, smooth Riemannian manifold of the topological type of the sphere, and specify a bounded, positive, measurable function ρ on Ω. Then there exists a conformal transformation $\Phi : \Omega \to S^2 \subset R^3$, embedded in the standard way as the unit sphere, such that*

$$\int_{S^2} \mathbf{x}\rho(\Phi^{-1}(\mathbf{x}))J d\hat{S} = \mathbf{0}. \tag{3}$$

Jacobian

Yet another "isoperimetric theorem," this time for λ_2.

For the trial function φ let's choose one of the Cartesian coordinates x,y,z on S^2, but "pull back" to Ω with the inverse of Hersch's conformal transformation. Let the resulting functions on Ω be called X,Y,Z. What do we know about X,Y,Z?

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1. **The functions X,Y,Z are orthogonal, because the functions x,y,z are orthogonal on S^2.**
   - **Note: The restrictions of x,y,z to S^2 are the spherical harmonics = eigenfunctions:**
     - $-\nabla^2 x = 2\,x,$
     - $-\nabla^2 y = 2\,y,$
     - $-\nabla^2 z = 2\,z,$

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1. The functions X,Y,Z are orthogonal.

2. **$X^2 + Y^2 + Z^2 = 1$, because $x^2 + y^2 + z^2 = 1$.**

Yet another "isoperimetric theorem," this time for λ_2.

For the trial function φ let's choose one of the Cartesian coordinates x,y,z on S^2, but "pull back" to Ω with the inverse of Hersch's conformal transformation. Let the resulting functions on Ω be called X,Y,Z. What do we know about X,Y,Z?

1. **The functions X,Y,Z are orthogonal.**

2. **$X^2 + Y^2 + Z^2 = 1$, because $x^2 + y^2 + z^2 = 1$.**

3. **Identifying now ρ with u_1,**

   $\langle X,u_1\rangle$ = $\int_{S^2} \mathbf{x} \rho(\Phi^{-1}(\mathbf{x})) J d\hat{S} = \mathbf{0}.$ **Likewise for Y, Z.**

Ready to roll with Rayleigh and Ritz:

Let's choose the trial function in

$$R(\zeta) := \frac{\int_\Omega |\nabla \zeta|^2 dS - \frac{1}{4}\int_\Omega (\kappa_2 - \kappa_1)^2 |\zeta|^2 dS}{\int_\Omega |\zeta|^2 dS}$$

as ζ =X, Y, or Z. Considering for example X, conformality implies that

$$\int_\Omega |\nabla X|^2 dS = \int_{S^2} |\nabla x|^2 d\hat{S} = \frac{8\pi}{3}$$

Ready to roll with Rayleigh and Ritz:

Observing that

$$a \leq \frac{b_j}{c_j}$$

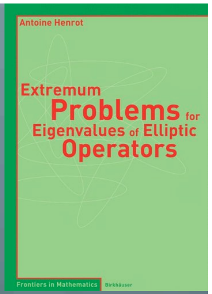
$\Longrightarrow$

$$a \leq \frac{\sum_j b_j}{\sum_j c_j}:$$

$$\lambda_2 \leq \frac{8\pi - \int_\Omega (\kappa_2 - \kappa_1)^2 dS}{\int_\Omega 1 dS}.$$

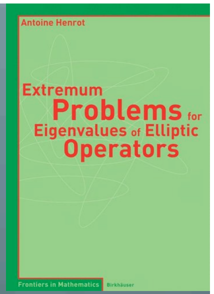
Equality iff sphere. Why?

Open problems in extreme spectral geometry



- Among all n-gons in the plane of a given area, does the regular n-gon produce the lowest λ_1?
- Is there in fact a minimizing domain (for fixed volume) for each λ_κ?
- In 2D, does the disk maximize $\lambda_2+\lambda$ if volume fixed? Does it maximize $(\lambda_2+\lambda_3)/\lambda_1$?
- Is $\lambda_{m+1}/\lambda_m \leq \lambda_2/\lambda_1$ for ball?

Open problems in extreme spectral geometry



- Can one characterize the minimizers of λ_k among *convex* domains of a given area/volume?

- Duality bounds. If K is a convex region, its *polar* dual is

  $K^* := \{x : x \cdot y \leq 1$ for all y in $K\}$. Find the best upper bound for $\lambda_1(K)\,\lambda_1(K^*)$ and the pair of regions that optimize it.


More open problems from somewhere deep in the Black Forest….

(Michael Loss) The Pólya Conjecture claims that the Weyl asymptotic formula provides a lower bound:

$$E_j \geq (2\pi)^2 (n/|S^{n-1}||\Omega|)^{2/n} j^{2/n}, \qquad j = 1, 2, 3, \ldots .$$

The conjecture remains open even for $j = 3$.

Consider eigenvalues of the Dirichlet Laplacian on a bounded convex domain $\Omega \subset \mathbb{R}^n$ with *convex* potential V:

$$\begin{cases} (-\Delta + V)u_j = \lambda_j u_j & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume $n \geq 1$. Notice the operator is written with $+V$, not $-V$ like in the previous section.

Van den Berg's Gap Conjecture is that

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{d^2}, \qquad d = \operatorname{diam}(\Omega),$$

with equality holding when $n = 1, V \equiv 0$. (In dimensions $n \geq 2$, the inequality should be strict, with equality holding only in the limit as the domain degenerates to an interval.)

A scheme for working on some of these optimal problems

Simplify matters by finding really efficient ways to *describe* the problem.

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For example, if we either assume a region is convex or have reason to think the optimal domain is convex, find a way to describe the objective with the support function.

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Simplify matters by finding really efficient ways to *describe* the problem.

For example, if we either assume a region is convex or have reason to think the optimal domain is convex, find a way to describe the objective with the support function.

If the region is at least *starlike*, we can describe things in terms of the gauge function.

Making new math in the math factory: First make prototypes and see if they work as hoped.

Making new math in the math factory: First make prototypes and see if they work as hoped.

For example, Antoine Henrot and I are exploring the direct use of the gauge and support functions. As test cases we are reproving Faber-Krahn by this method and exploring upper variational bounds.

 2 Feb 09

Here is a convex geometric analysis way to prove Faber-Krahn for star-shaped domains, for which there is a gauge function. I do it in two dimensions, assuming that for a polar angle θ, the boundary of K is given by $r = 1/u(\theta)$. (This was a form suggested by the observation that convexity $\Leftrightarrow$ $u''+u \geq 0$ and that u is related to the gauge function.) The gauge function is

$$g(r,\theta) = r\,u(\theta) \quad (\text{so } g = 1 \text{ on } \partial K).$$

Let's change variables from (r,θ) to (g,θ). The metric tensor in (g,θ) is

$$g_{ij} = \frac{1}{u^2}\begin{pmatrix} 1 & -g u'/u \\ -g u'/u & g^2(1+(u'/u)^2) \end{pmatrix}, \quad \det g_{ij} = \frac{g^2}{u^4},$$
$$dA = \frac{g}{u^2}\,dg\,d\theta.$$

$$g^{ij} = u^2 \begin{pmatrix} (1+w^2) & w/g \\ w/g & 1/g^2 \end{pmatrix} \quad \text{where } w = u'/u$$

The energy form can be written

$$E(\varphi) = \int_0^{2\pi}\int_0^1 \left\{ \varphi_g^2 + \left(w\varphi_g + \frac{1}{g}\varphi_\theta\right)^2 \right\} g\,dg\,d\theta.$$

The area is fixed as

$$A = \frac{1}{2}\int u^{-2}\,d\theta = \frac{1}{2}\int e^{-2\int w}\,d\theta.$$

Observe that the integral can be arranged to equal A by a choice of the constant of integration.

$$\lambda_1^* = \min_W \min_{\|\varphi\|=1} E(\varphi) = \min_\varphi \min_W E(\varphi).$$

The first minimization when $E(\varphi)$ is written this way is

$$\int_0^1 \varphi_g^2\,g\,dg + W^2\int_0^1 \varphi_g^2\,g\,dg + 2W\int_0^1 \varphi_g\varphi_\theta\,dg + \int_0^1 \varphi_\theta^2\,\frac{1}{g}\,dg$$

$$\geq \int_0^1\left(\varphi_g^2\,g\,dg + \varphi_\theta^2\,\frac{1}{g}\,dg\right) + \frac{\left(\int_0^1 \varphi_g\varphi_\theta\,dg\right)^2}{\int \varphi_g^2\,g\,dg}$$

This in turn

$$\geq \int_0^1 \varphi_g^2\,g\,dg.$$

Thus if we minimize over φ we are bounded below by the same quantity that appears in the radially-symmetric subspace $(g \to r)$

How to argue away the u_* in

$$\star \quad \frac{\int_0^{2\pi}\int_0^1 \varphi_g^2\,g\,dg\,\frac{1}{u_*^2}\,d\theta}{\int_0^{2\pi}\int_0^1 \varphi^2\,g\,dg\,\frac{1}{u^2}\,d\theta} \quad ?$$

Perhaps argue that $\int_0^1 \varphi_g^2\,g\,dg$ is equivalent to the quadratic form of the Laplacian projected onto the radial subspace and therefore $\geq \lambda_1^* \int_0^1 \varphi^2\,g\,dg$ at each fixed θ. Therefore

$$\star \geq \lambda_1^* \cdot 1$$

(Seems ok. + but what about the volume element?)

If we integrate $E(\varphi)$ by parts we get:

$$E(\varphi) = -\int_0^{2\pi}\int_0^1 \varphi\left((1+w^2)\left(\varphi_{gg} + \frac{1}{g}\varphi_g\right) + \frac{2w}{g}\varphi_{g\theta} + \frac{\varphi_{\theta\theta}}{g^2}\right) g\,dg\,d\theta$$

(For the true Laplacian, multiply and divide by $u^2 = e^{2\int w}$.)

$\to$ The minimum occurs for φ_* indep. of θ, and therefore

$$\int_0^{2\pi}\int_0^1 \varphi_g^2\,g\,dg\,d\theta \geq \lambda_1^* \cdot \int_0^{2\pi}\int_0^1 \varphi_*^2\,g\,dg\,d\theta$$

$$= \lambda_1^* \cdot 2\pi \cdot \int_0^1 \varphi_*^2\,g\,dg = \lambda_1^*\,\frac{2\pi}{A}\int_0^{2\pi}\int_0^1 \varphi_*^2\,g\,dg\,\frac{d\theta}{u^2}$$

$$=$$

17 Apr. 09. Antoine and I checked out the bound using $1-g$ as a test function. It turns out that the bound using the support function in Freitas - Krejčiřík is slightly better for rectangles, for example. Later I tried $1-g^2$, but it gave the same results for the disk as $1-g$. (!). Therefore, I thought that a good trial function would be

$\varphi = J_0(j_{01} g)$. Note $\|\varphi\|^2 = \int J_0^2(j_{01} g) g\,dg \cdot \int \frac{d\theta}{u^2}$

$\doteq 0.134757 \int \frac{d\theta}{u^2}$

$= 0.134757 \cdot 2|\Omega|$

Thus,

$$\nabla\varphi = j_{01} J_0'(j_{01} g) \nabla g$$

$$= j_{01} J_0'(j_{01} g)\left(u\hat{e}_r + u'\hat{e}_\theta\right)$$

$$\int |\nabla\varphi|^2 da = j_{01}^2 \int_0^{2\pi}\int_0^1 |J_0'(j_{01} g)|^2 (u^2)\left(1+\left(\frac{u'}{u}\right)^2\right) da$$

with $da = \frac{g}{u^2} dg\,d\theta$,

$$\int(\nabla\varphi)^2 = j_{01}^2 \int_0^1 |J_0'(j_{01} g)|^2 g\,dg \cdot 2\pi + j_{01}^2 \int_0^1 |J_0'(j_{01} g)|^2 g\,dg \cdot \int_0^{2\pi} \left(\frac{u'}{u}\right)^2 d\theta$$

$$\frac{\int|\nabla\varphi|^2}{\int|\varphi|^2} = j_{01}^2 \frac{\pi}{|\Omega|} + \frac{j_{01}^2 \int_0^1 |J_0'(j_{01} g)|^2 g\,dg \int_0^{2\pi} \left(\frac{u'}{u}\right)^2 d\theta}{\int_0^1 (J_0(j_{01} g))^2 g\,dg \cdot 2|\Omega|}$$

There is still the matter of optimizing over the origin.

By dropping the last term, we get a version of Faber-Krahn, or at least an upper bound not much bigger. If we keep it, we have, for $|\Omega| = \pi$

$$\boxed{\lambda_1 \leq j_{01}^2\left(1 + \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{u'}{u}\right)^2 d\theta\right)}$$

I guess this is a good inequality, true for star-shaped domains with boundary piecewise differentiable, or in Sobolev H^1 space. Note if $\partial K = \{r = r(\theta)\}$, $\left(\frac{u'}{u}\right)^2 = \left(\frac{r'}{r}\right)^2$.

Higher dimensions? Let's just call the ground state of the ball $\Phi_*(r)$, $0 \leq r \leq 1$.

Then $\nabla(\Phi_* \circ g) = \Phi_*'(g)\nabla g$

where g is of the form $g = r u(\omega)$. Thus

$$|\nabla(\Phi_* \circ g)|^2 = (\Phi_*'(g))^2\left(u^2(\omega) + |\nabla_0 u(\omega)|^2\right)$$

$$\frac{\int|\nabla(\Phi_* \circ g)|^2 dV}{\int(\Phi_* \circ g)^2 dV} =$$

Some loopy problems

$$\mathbf{H(g) = -\frac{d^2}{ds^2} + g\kappa^2}$$

s = arclength, κ = curvature, and g = a "coupling constant"

Isoperimetric theorems for $-\,d^2/ds^2 + g\,\kappa^2$

I. One dimension

$$-\frac{d^2}{ds^2} + g\kappa^2$$

Ω - curve.

A. Ω infinitely long, asymptotically straight

$g<0$

$\lambda_1 < 0$ unless Ω is a line

Duclos - Exner

B. Ω closed, say $|\Omega| = 1$

(i) $g \leq 0$,

λ_1 uniquely maximized by ○

Duclos – Exner

(ii)

$g = -1$

λ_2 uniquely maximized by ○

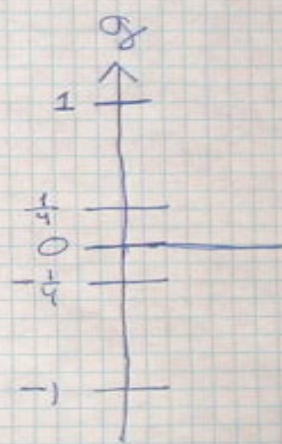
Harrell – Loss

(iii)

$0 \leq g \leq \frac{1}{4}$

λ_1 uniquely minimized by ○

Exner – Harrell – Loss



$\mathcal{C}$ - length 2π, closed

$$-\frac{d^2}{ds^2} + g\,\kappa^2 \quad , \quad g \leq 0$$

$\Rightarrow$ $\lambda_1 \leq 0$, <u>max</u>imized by circle.

<u>Proof</u> Rayleigh-Ritz $\Rightarrow$

$$\lambda_1 \int 1^2 ds \leq \int ((1')^2 + g\kappa^2 1^2)\, ds$$

$$2\pi \lambda_1 = -|g| \int \kappa^2 ds = -|g| \frac{\int \kappa^2 ds \int 1^2 ds}{2\pi}$$

Cauchy-Schwarz $$\leq -\frac{|g|}{2\pi} \left(\int \kappa \cdot 1 \right)^2 = -|g| \cdot 2\pi$$

$$\lambda_1 \leq -|g|$$

Equality iff $\kappa = \text{cst.}$ a.e. (circle).

Minimality when g ≤ 1/4.

$$\mathbf{H(g) = -\frac{d^2}{ds^2} + g\kappa^2}$$

If $0 < g \leq 1/4$, the unique curve minimizing λ_1 is the circle (Exner-Harrell-Loss).

“The loop problem”

$$\mathbf{H(g) = -\frac{d^2}{ds^2} + g\kappa^2}$$

If $0 < g \leq 1/4$, the unique curve minimizing λ_1 is the circle (Exner-Harrell-Loss). If $g > 1$, no longer true.

**What happens in between? OPEN.

The End

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