Eigenvalue distributions and the structure of graphs

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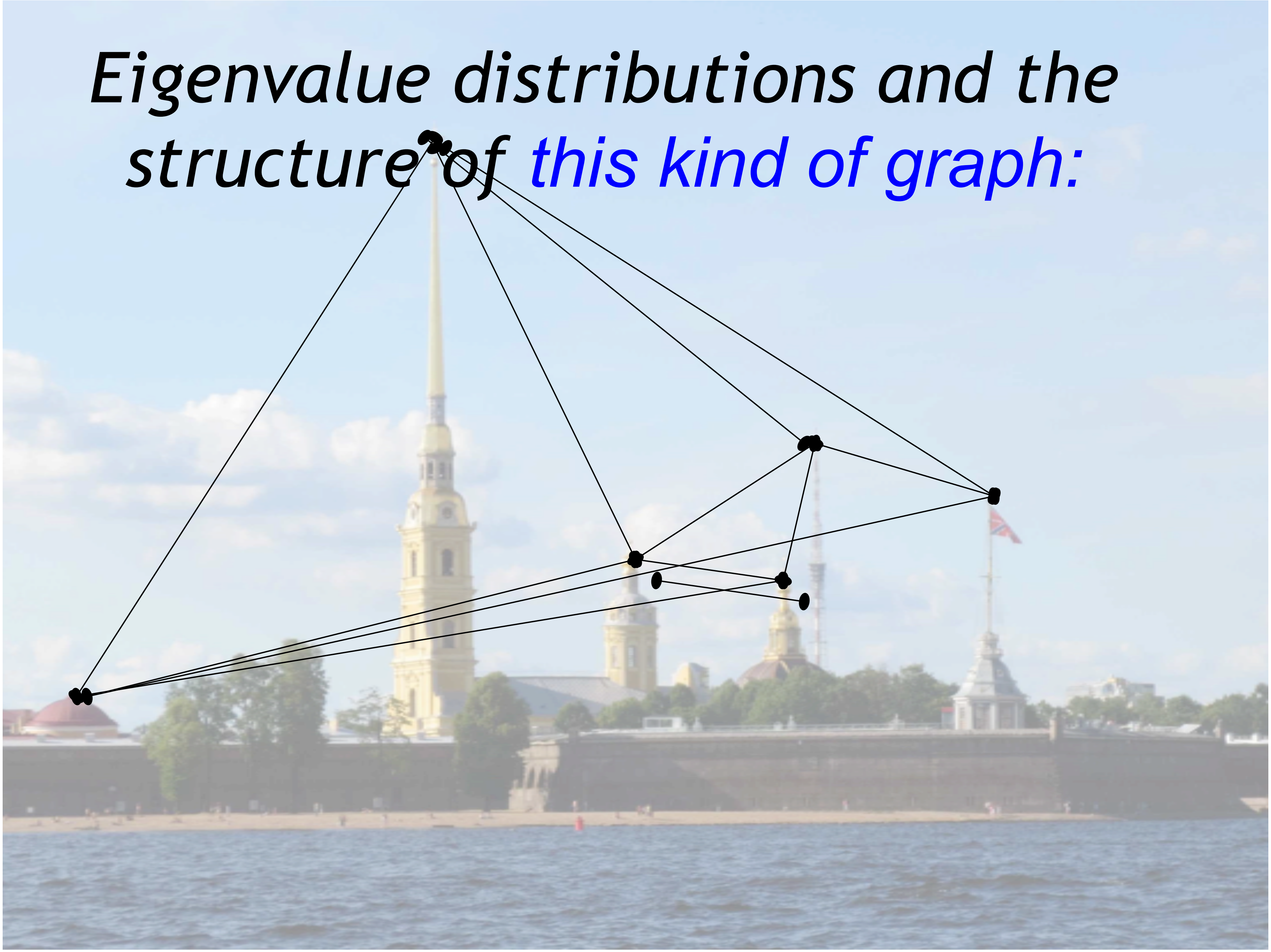
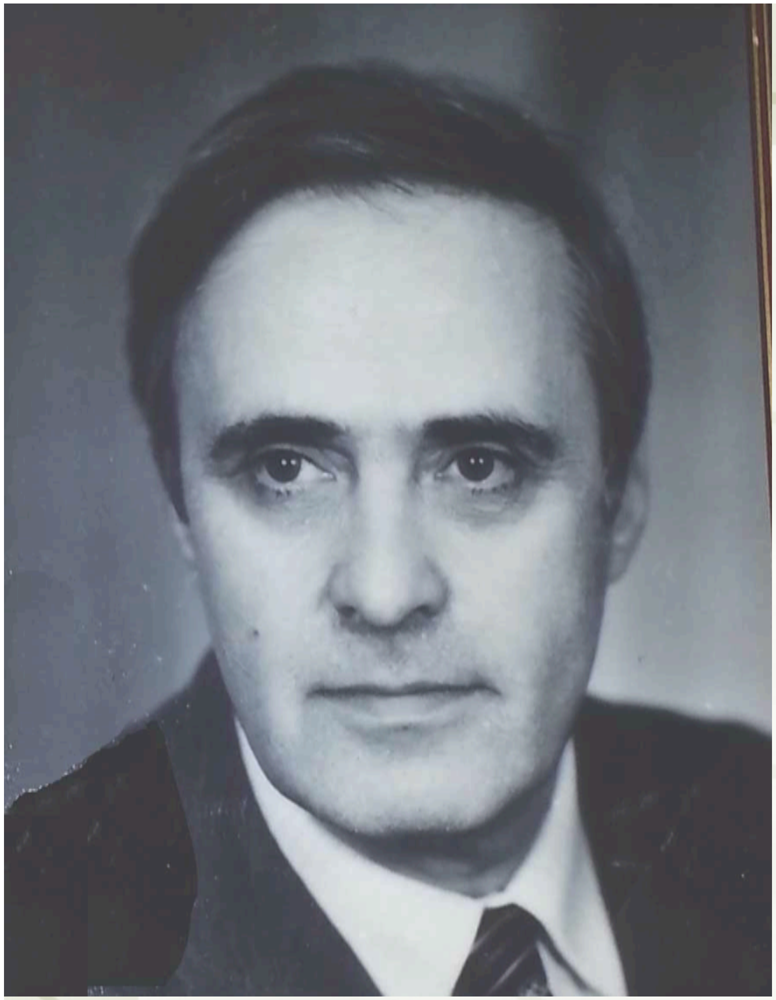
Eigenvalue distributions and the structure of this kind of graph:





Abstract

- We consider the spectra of three self-adjoint matrices associated with a combinatorial graph, viz., the adjacency matrix A, the graph Laplacian $H=-\Delta$, and the normalized graph Laplacian $\mathcal{L}$. Using a) variational techniques, and b) identities for traces of operators and Chebyshev's inequality, we present some bounds on gaps, sums, Riesz means, and the statistical distribution of eigenvalues of these operators, and relate them to the structure of the graph
- This is a preliminary report on recent work with J. Stubbe of ÉPFL.

The essential message of this seminar

- It is well known that the largest and smallest eigenvalues, and some other spectral properties, such as determinants, satisfy simple inequalities and provide information about the structure of a graph.
- It will be shown that statistical properties of spectra (means, variance) also satisfy certain inequalities, which provide information about the structure of a graph.

A nanotutorial on graph spectra

- A graph on n vertices is in 1-1 correspondence with an an n by n adjacency matrix A, with $a_{ij} = 1$ when i and j are connected, otherwise 0.

Generic assumptions: connected, not directed, finite, at most one edge between vertices, no self-connection...

A nanotutorial on graph spectra

- A graph on n vertices is in 1-1 correspondence with an an n by n adjacency matrix A, with $a_{ij} = 1$ when i and j are connected, otherwise 0.
  - How is the structure of the graph reflected in the spectrum of A?
  - What sequences of numbers might be spectra of A?

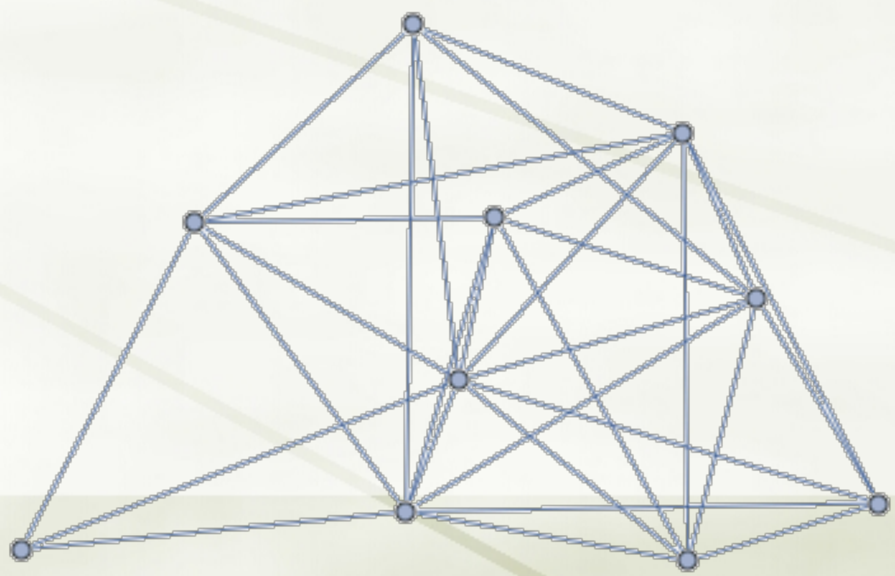
Some generic graph

In[160]:= `MatrixForm[EX1]`

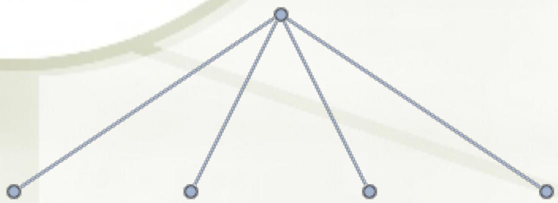
Out[160]//MatrixForm=

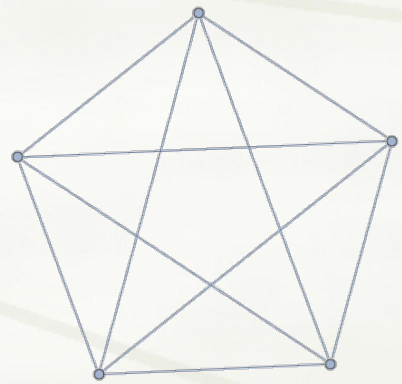
$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

`AdjacencyGraph[EX1]`



(The usual suspects)



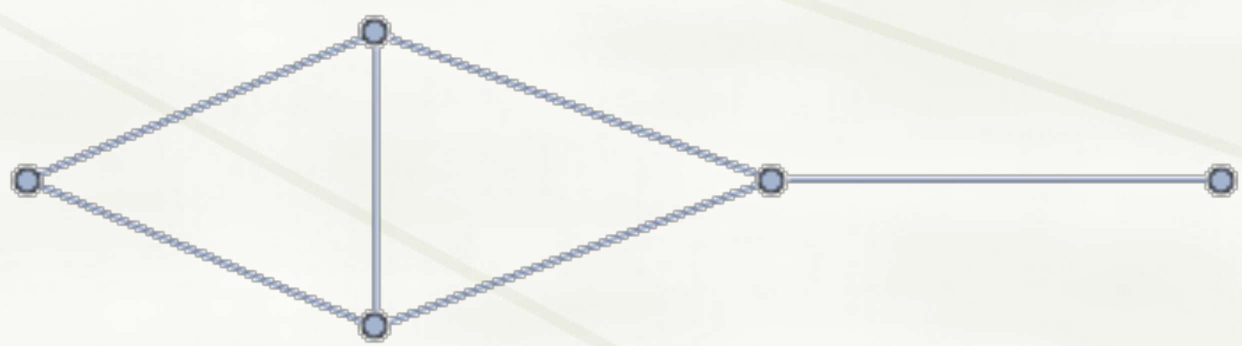


$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

A nanotutorial on graph spectra

In[163]:= TypEX0 = $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$;

In[164]:= AdjacencyGraph[TypEX0]

Out[164]= 

What are the quantitative ways to describe the structure of graphs?

- Disconnectability (how many edges or vertices must be removed)
- Colorability
- Numbers of triangles, spanning trees, and other simple subgraphs.(n-cycles, cliques, matchings, ...)
- Moments of degrees (or l^p-means of the number of neighbors)

A nanotutorial on graph spectra

- The *graph Laplacian* is a matrix that compares values of a function at a vertex with the average of its values at the neighbors.

$$H := -\Delta := \mathrm{Deg} - A, \text{ where}$$

$\mathrm{Deg} := \mathrm{diag}(d_v)$, $d_v :=$ # neighbors of v.

A nanotutorial on graph spectra

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  $$H := -\Delta := \mathrm{Deg} - A, \text{ where}$$

  Deg := diag(d_v), d_v := # neighbors of v.
- The quadratic form is

$$f \to \frac{1}{2}\sum_{u}\sum_{v\sim u}|f_u - f_v|^2$$

A nanotutorial on graph spectra

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- How is the structure of the graph reflected in the spectrum of $-\Delta$?
- What sequences of numbers might be spectra of $-\Delta$?

A nanotutorial on graph spectra

- There is also a normalized graph Laplacian, favored by Fan Chung

$$C := Deg^{-\frac{1}{2}} H Deg^{-\frac{1}{2}}$$

A nanotutorial on graph spectra

- There is also a normalized graph Laplacian, favored by Fan Chung.
- The spectra of the three operators are trivially related if the graph is regular (all degrees equal), but otherwise not.

The most basic spectral facts

- Adding edges is equivalent to $A \to A + A_E$.
- The spectrum of A allows one to count "spanning subgraphs."
- It easily determines whether the graph has 2 colors. "bipartite"
- The max eigenvalue is ≤ the max degree.
- There is an interlacing theorem when an edge is added.

The most basic spectral facts

- H ≥ 0 and H **1** = 0 **1**. (Like Neumann)
- Taking unions of disjoint edge sets,

$$H_{G_1 \cup G_2} = H_{G_1} + H_{G_2}$$

- This implies a relation between the spectra of a graph and of its edge complement, and various useful simple inequalities.
- The spectrum determines the number of spanning trees (classic thm of Kirchhoff)
- There is an interlacing theorem when an edge is added

The most basic spectral facts

- For none of the operators on graphs is it known which precise sets of eigenvalues are feasible spectra.

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The most basic spectral facts

- For none of the operators on graphs is it known which precise sets of eigenvalues are feasible spectra.
- Examples of nonequivalent isospectral graphs are known (and not too tricky)
  - But isospectral with respect to two of the operators?
- Eigenfunctions can sometimes be supported on small subsets.

A nanotutorial on graph spectra

- There are books on graph spectra by
  - 金芳蓉
  - Cvetković et al.
  - Bıyıkoğlu et al.

Three good philosophies for understanding graph spectra

- Make variational estimates.

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Three good philosophies for understanding graph spectra

- Make variational estimates.
- Exploit algebraic identities for traces anddeterminants.
- Use statistical identities, the coefficients of which connect to graph structures.

Variational bounds on graph spectra

Variational bounds on graph spectra

- Inequalities that arise from min-max and good choices of trial functions.

- For example, Fiedler showed in 1973 that for the graph Laplacian
  ($0 = \lambda_0 < \lambda_1 \le \ldots \le \lambda_{n-1} \le n$)

$$\lambda_1 \le \frac{n}{n-1} \min_k d_k, \quad \frac{n}{n-1} \max_k d_k \le \lambda_{n-1}$$

Variational bounds on graph spectra

- Inequalities that arise from min-max and good choices of trial functions.

Let M be a selfadjoint operator on a Hilbert space of dimension n, with associated eigenvalues $\mu_0 \leq \cdots \leq \mu_{n-1}$. Then, for any orthonormal basis $\{\phi_i\}$ and any $1 \leq k \leq n-1$, we get

$$\sum_{i=0}^{k-1} \mu_i \leq \sum_{i=0}^{k-1} \langle \phi_i, M\phi_i \rangle.$$

Variational bounds on graph spectra

- There are some good choices of trial functions that appear not to have been exploited before.

$$H_p := \left(\begin{array}{cccc|cccc} p & 0 & \dots & 0 & -1 & -1 & \dots & -1 \\ 0 & p & \dots & 0 & -1 & -1 & \dots & -1 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & p & -1 & -1 & \dots & -1 \\ \hline -1 & -1 & \dots & -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & \dots & -1 & -1 & n-1 & & -1 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ -1 & -1 & \dots & -1 & -1 & -1 & \dots & n-1 \end{array}\right). \tag{2.2}$$

Remark 1 *In particular, H_1 is the Laplacian of a star graph, while H_{n-1} is the Laplacian of a complete graph. For future purposes we observe that*

$$tr(H_p) = p(2n-p-1), \quad tr(H_p^2) = p(n^2+pn-p^2-p) \tag{2.3}$$

Building a variational estimate for an arbitrary graph from the eigenvectors of this family of graphs leads to an extension of a result of Fiedler [7], as we next demonstrate.

Proposition 2.1 *(Spectral analysis of H_p.) Let*

$$\overline{\mathbf{e}}_k := \frac{1}{\sqrt{k(k+1)}}\Big(k\mathbf{e}_{k+1} - \sum_{j=1}^{k} \mathbf{e}_j\Big), \tag{2.4}$$

where $\mathbf{e}_j, j = 1\dots,n$ denote the canonical orthonormal basis vectors of $\mathbb{R}^n$. Then $\{\overline{\mathbf{e}}_k, k = 0\dots,n-1\}$ is an orthonormal basis of $\mathbb{R}^n$. For each $k = 1,\dots,n-p-1$, $\overline{\mathbf{e}}_k$ is an eigenvector of H_p with corresponding eigenvalue p, and for each $k = n-p,\dots,n-1$, $\overline{\mathbf{e}}_k$ is an eigenvector of H_p with corresponding eigenvalue n.

Variational bounds on graph spectra

$$\sum_{\ell=1}^{L} \lambda_\ell \leq \frac{(n-L+1)\sum_{k=n-L+1}^{n} d_k - 1}{n-L}$$

- (where the degrees are in decreasing order)

$$\sum_{\ell}^{L} \lambda_\ell \leq \frac{L\sum_{k}^{L+1} d_k - 1}{L+1}$$

- optimal for the complete and star graphs

Variational bounds on graph spectra

Alternative for $\lambda_1 + \lambda_2$:

$$\lambda_1 + \lambda_2 \leq \frac{2E}{n-2} + \frac{n(n-3)}{(n-1)(n-2)} \min_k d_k$$

Variational bounds on graph spectra

- Generalization of Fiedler:

For any $L = 1, \ldots, n-1$ *we get*

$$\sum_{i=1}^{L} \lambda_i \leq \frac{L}{L+1} \sum_{i=1}^{L+1} h_{ii} - \frac{1}{L+1} \sum_{\alpha=1}^{L+1} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{L+1} h_{\alpha\beta} \leq \sum_{i=N-L+1}^{N} \lambda_i$$

$$\sum_{i=1}^{L} \lambda_i \leq \frac{n-L+1}{n-L} \sum_{i=n-L+1}^{n} h_{ii} - \frac{1}{n-L} \sum_{\alpha=n-L+1}^{n} \sum_{\substack{\beta=n-L+1 \\ \beta\neq\alpha}}^{n} h_{\alpha\beta} \leq \sum_{i=N-L+1}^{N} \lambda_i.$$

Variational bounds on graph spectra

- In 1992 Pawel Kröger found a variational argument for the Neumann counterpart to Berezin-Li-Yau, i.e. a Weyl-sharp upper bounds on sums of the eigenvalues of the Neumann Laplacian

$$\sum_{\ell}^{L} \lambda_\ell$$

Variational bounds on graph spectra

- The graph Laplacian should be thought of as Neumann, rather than Dirichlet. By making an abstract version of Kröger’s argument we can derive interesting upper bounds on sums of eigenvalues of H, A, and C, and some other inequalities relating eigenvalues to graph structures.

An abstract version of Kröger's inequality

Lemma 4 *Consider a self-adjoint operator M on a Hilbert space $\mathcal{H}$, with ordered, entirely discrete spectrum $-\infty < \mu_0 \le \mu_1 \le \dots$ and corresponding normalized eigenvectors $\{\phi_k\}$. Let f_z be a family of vectors in $D(H)$ indexed by a variable z ranging over a measure space $(\mathfrak{M}, \Sigma, m)$. Suppose that $\mathfrak{M}_0$ is a subset of $\mathfrak{M}$. Then:*

$$\mu_k\left(\int_{\mathfrak{M}_0} \langle f_z, f_z\rangle \, dm - \int_{\mathfrak{M}} \sum_{j=0}^{k-1} |\langle f_z, \phi_j\rangle|^2 \, dm \right) \le \int_{\mathfrak{M}_0} \langle H f_z, f_z\rangle dm - \int_{\mathfrak{M}} \sum_{j=0}^{k-1} \mu_j |\langle f_z, \phi_j\rangle|^2 \, dm, \tag{2.15}$$

provided that the integrals converge.

Proof of abstract Kröger inequality

By the variational principle (2.1),

$$\mu_k\Big(\langle f, f\rangle - \langle P_{k-1}f, P_{k-1}f\rangle\Big) \leq \langle Mf, f\rangle - \langle MP_{k-1}f, P_{k-1}f\rangle.$$

Proof. By integrating (2.14),

$$\mu_k \int_{\mathfrak{M}_0} \left(\langle f_z, f_z\rangle - \langle P_{k-1}f, P_{k-1}f_z\rangle\right) dm \leq \int_{\mathfrak{M}_0} \langle Hf_z, f_z\rangle\, dm - \int_{\mathfrak{M}_0} \langle HP_{k-1}f_z, P_{k-1}f_z\rangle\, dm, \tag{2.16}$$

or

$$\mu_k \int_{\mathfrak{M}_0} \left(\langle f_z, f_z\rangle - \sum_{j=0}^{k-1} |\langle f_z, \phi_j\rangle|^2\right) dm \leq \int_{\mathfrak{M}_0} \langle Hf_z, f_z\rangle\, dm - \int_{\mathfrak{M}_0} \sum_{j=0}^{k-1} \mu_j |\langle f_z, \phi_j\rangle|^2\, dm. \tag{2.17}$$

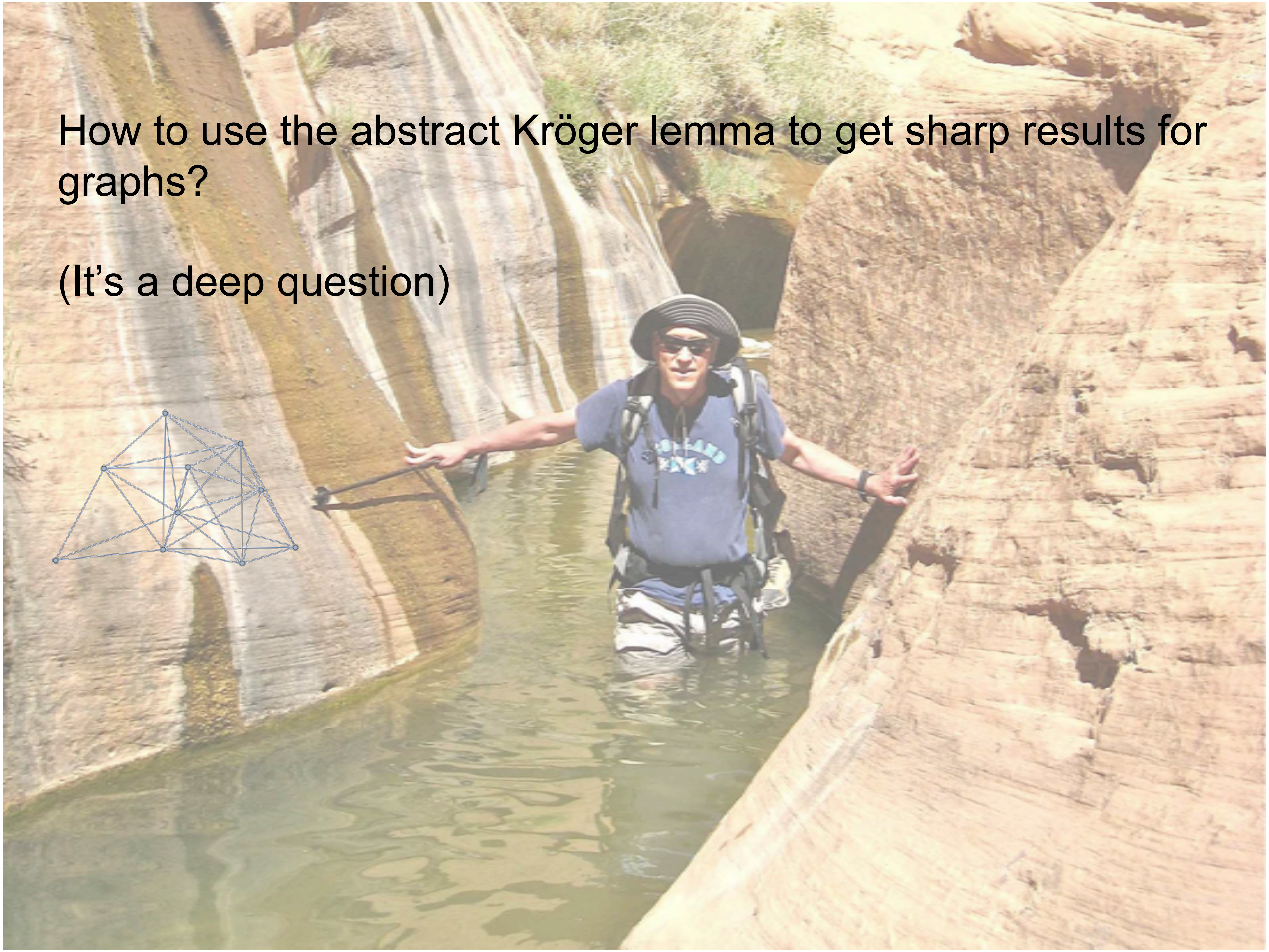
Since μ_k is larger than or equal to any weighted average of $\mu_1 \ldots \mu_{k-1}$, we add to (2.17) the inequality

$$-\mu_k \int_{\mathfrak{M}\setminus\mathfrak{M}_0} \left(\sum_{j=0}^{k-1} |\langle f_z, \phi_j\rangle|^2\right) dm \leq -\int_{\mathfrak{M}\setminus\mathfrak{M}_0} \sum_{j=0}^{k-1} \mu_j |\langle f_z, \phi_j\rangle|^2\, dm, \tag{2.18}$$

and obtain the claim. $\square$

How to use the abstract Kröger lemma to get sharp results for graphs?

(It’s a deep question)



Corollary 6 *Suppose that* G *is a finite subgraph of* Ω^ν. *Then for* $k \geq 2$ *the eigenvalues of the graph Laplacian* H_G *satisfy*

$$\sum_{j=1}^{k-1} \lambda_j \leq 2\mathcal{E}\left(1 - \frac{\sin((k/n)^{1/\nu}\pi)}{(k/n)^{1/\nu}\pi}\right)\frac{k}{n}, \tag{2.17}$$

where $\mathcal{E}$ *denotes the number of edges of* G.

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where $\mathcal{E}$ denotes the number of edges of G.

Remark 2.2 *In particular, it is true independently of dimension that*

$$\sum_{j=1}^{k-1} \lambda_j \leq \frac{2\mathcal{E}k}{n},$$

which becomes a standard equality when $k = n$. In the complementary situation where $k << n$ the upper bound is

$$\sim \frac{\pi^2\mathcal{E}}{3}\left(\frac{k}{n}\right)^{1+\frac{2}{\nu}},$$

which has the form of the Weyl law for Laplacians on domains $\Omega \subset \mathbb{R}^\nu$.

$$M = [-\pi, \pi]^\nu$$

$$M_0 := [-a\pi, a\pi]^\nu$$

$$f_{\mathbf{x}} = \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$\hat{\phi}(\mathbf{x}) := \sum_{\mathbf{k}\in G} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\, \phi_{\mathbf{k}}, \qquad \phi_{\mathbf{k}} = \frac{1}{(2\pi)^\nu} \int_{[-\pi,\pi]^\nu} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\, \hat{\phi}(\mathbf{x})$$

$$\left\langle H\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}}, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} \right\rangle = \frac{1}{2} \sum_{\mathbf{k}\in G} \sum_{\mathbf{p}\sim\mathbf{k}} |\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} - \mathrm{e}^{i\mathbf{p}\cdot\mathbf{z}} \,|^2$$

$$\left\langle H\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}}, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} \right\rangle = \frac{1}{2} \sum_{\mathbf{k}\in G} \sum_{\mathbf{p}\sim\mathbf{k}} |\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} - \mathrm{e}^{i\mathbf{p}\cdot\mathbf{z}}\,|^2$$

$|\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} - \mathrm{e}^{i\mathbf{p}\cdot\mathbf{z}}\,|^2$ simplifies to $|e^{\pm i z_q} - 1|^2 = 4\sin^2\left(\frac{z_q}{2}\right)$

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$$A_0 = 2(a\pi - \sin(a\pi))(2a\pi)^{\nu-1}\sum_{\mathbf{k}\in G} d_k = (2a\pi)^{\nu} 2\left(1 - \frac{\sin(a\pi)}{a\pi}\right)\mathcal{E}$$

$$\left\langle H\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}}, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} \right\rangle = \frac{1}{2} \sum_{\mathbf{k}\in G} \sum_{\mathbf{p}\sim\mathbf{k}} |\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{z}} - \mathrm{e}^{i\mathbf{p}\cdot\mathbf{z}} \,|^2$$

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$$\int_{[-\pi,\pi]^\nu} |\left\langle \mathrm{e}^{i\mathbf{k}\dot{\mathbf{x}}}, \phi_j \right\rangle|^2 = (2\pi)^\nu \|\phi_j\|^2 = (2\pi)^\nu$$

Meanwhile, on the left, we need

$$n(2a\pi)^\nu - k(2\pi)^\nu \geq 0,$$

$$\text{so } a^\nu \to k/n$$

Giving

$$\sum_{j=1}^{k-1} \lambda_j \leq 2\mathcal{E}\left(1 - \frac{\sin((k/n)^{1/\nu}\pi)}{(k/n)^{1/\nu}\pi}\right)\frac{k}{n}$$

Kröger method for H^2 or other $f(H)$?

Kröger method for H^2 or other $f(H)$?

$$\sum_{j=1}^{k-1} \lambda_j^2 \le a^\nu \left[\left(1-\left(\frac{\sin(a\pi)}{a\pi}\right)^2\right)\sum_{\mathbf{p}\in G} d_{\mathbf{p}} \right.$$
$$\left. + \left(1-2\frac{\sin(a\pi)}{a\pi}+\left(\frac{\sin(a\pi)}{a\pi}\right)^2\right)\sum_{\mathbf{p}\in G} d_{\mathbf{p}}^2 + \left(\frac{\sin(2a\pi)}{a\pi} - 2\left(\frac{\sin(a\pi)}{a\pi}\right)^2\right)\sum_{\mathbf{p}\in G}\hat{d}_{\mathbf{p}}\right]$$

again, $a^\nu \to k/n$, but this is not Weyl-correct!

Extensions to traces of concave functions of λ_j and to partition functions

Extensions to traces of concave functions of λ_j and to partition functions

Lemma 5 *(Karamata-Ostrowski; e.g., see [1], §28.) Let two nondecreasing ordered sequences of real numbers $\{mu_j\}$ and $\{M_j\}$, $j = 0, \ldots, n-1$, satisfy*

$$\sum_{j=0}^{k-1} \mu_j \leq \sum_{j=0}^{k-1} M_j \tag{2.20}$$

for each k. Then for any nondecreasing concave function $\phi(x)$ and each k,

$$\sum_{j=0}^{k-1} \phi(\mu_j) \leq \sum_{j=0}^{k-1} \phi(M_j).$$

Similarly, for any nonincreasing convex function $\psi(x)$,

$$\sum_{j=0}^{k-1} \psi(\mu_j) \geq \sum_{j=0}^{k-1} \psi(M_j).$$

Variational bounds on graph spectra

Another way to apply the abstract Kröger lemma to graphs is to let $\mathcal{M}$ be the set of pairs of vertices. The reason is that the complete graph has a superbasis of nontrivial eigenfunctions consisting of functions equal to 1 on one vertex, -1 on a second, and 0 everywhere else. Let these functions be h_z, where z is a vertex pair.

Variational bounds on graph spectra

Two facts are easily seen:

1. For vectors of mean 0 (orthogonal to $\phi_0 = 1$),

$$\sum_{\text{all pairs}} |\langle h_{uv}, f\rangle|^2 = 2\|f\|^2(n-1).$$

2. $$\langle H h_{uv}, h_{uv}\rangle = d_u + d_v + 2a_{uv}$$

Variational bounds on graph spectra

It follows from Kröger's lemma that

$$\sum_{j \leq L} \lambda_j \leq \frac{1}{2n} \min_{\text{choices of nL pairs}} \sum_{uv} (d_u + d_v + 2a_{uv})$$

Variational bounds on graph spectra

- Extensions to renormalized Laplacian

Corollary 8 *Let G be any finite graph on n vertices, and let M_0 be any set of p pairs of vertices $\{u, v\}$ with $\sum_{M_0} d_u + d_v \geq (k-1)2\mathcal{E}$. Then the eigenvalues of the renormalized Laplacian C_G satisfy*

$$\sum_{j=1}^{k-1} \chi_j \leq \sum_{M_0} (2 + d_u + d_v).$$

Variational bounds on graph spectra

- How about the adjacency matrix?

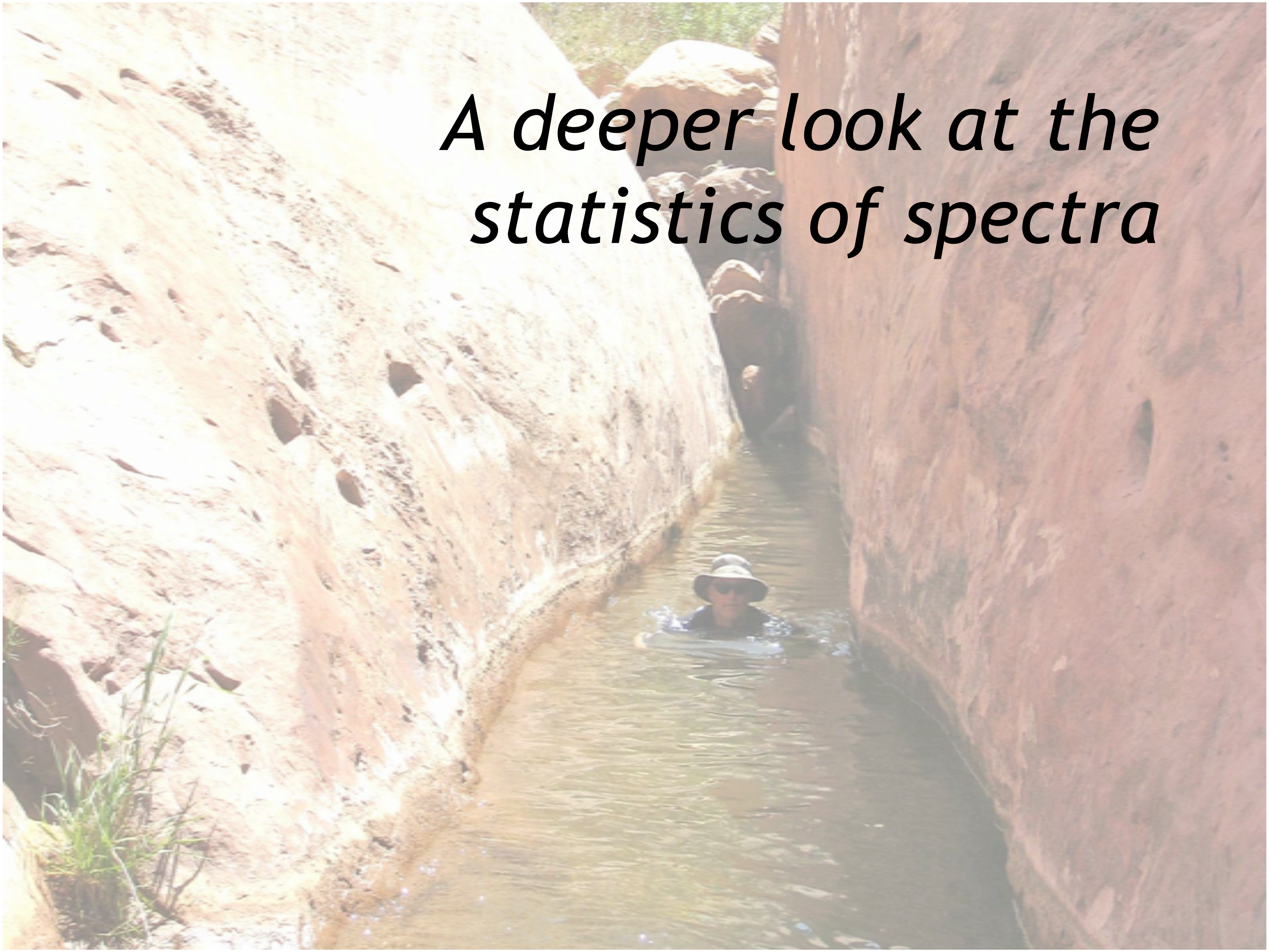
The analogous result for the adjacency matrix reads as follows

Corollary 9 *Let G be any finite connected graph on n vertices. Then for $1 \leq k < n$, the eigenvalues $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{n-1}$ of the adjacency matrix A_G satisfy*

$$\sum_{j=0}^{n-k-1} \alpha_j \geq k,$$

$$\sum_{j=n-k}^{n-1} \alpha_j \leq -k. \tag{2.30}$$

A deeper look at the statistics of spectra



Pafnuty Chebyshev

From Wikipedia, the free encyclopedia

"Chebyshev" redirects here. For other uses, see Chebyshev (disambiguation).

Pafnuty Lvovich Chebyshev (Russian: **Пафну́тий Льво́вич Чебышёв**, IPA: [pɐfˈnutʲɪj ˈlʲvovʲɪtɕ tɕɪbɨˈʂof]) (May 16 [O.S. May 4] 1821 – December 8 [O.S. November 26] 1894)[1] was a Russian mathematician. His name can be alternatively transliterated as **Chebychev** (English transliteration), **Chebysheff** (English), **Chebyshov** (English), **Tchebychev** (French) or **Tchebycheff** (French), or **Tschebyschev** (German) or **Tschebyscheff** (German) or **Tschebyschow** (German).

Pafnuty Chebyshev	
	
Pafnuty Lvovich Chebyshev	
Born	May 16, 1821 Borovsk, Kaluga, Russian Empire
Died	December 8, 1894 (aged 73) St Petersburg, Russian Empire
Nationality	Russian
Fields	Mathematician
Institutions	St Petersburg University
Alma mater	Moscow University
Doctoral advisor	Nikolai Brashman
Doctoral students	Dmitry Grave Aleksandr Korkin Aleksandr Lyapunov Andrey Markov Vladimir Andreevich Markov Konstantin Posse
Known for	Mechanics and analytical geometry
Notable awards	Demidov Prize (1849)

Contents [hide]

Biography [edit]

One of nine children, he was born in the central Russian village of Okatovo near Borovsk, to Agrafena Ivanova Pozniakova and Lev Pavlovich Chebyshev. His father fought as an officer against Napoleon's invading army.

He was originally home-schooled by his mother and his cousin Avdotia Kvintillianova Soukhareva. He learned French early in life, which later helped him communicate with other mathematicians. A stunted leg prevented him from playing with other children, leading him to concentrate on studying instead.

Later he studied at Moscow University obtaining his degree in 1841.

He was a student of Nikolai Brashman. His own most illustrious student was Andrey Markov, although Alexandr Lyapunov is also famous for the method that bears his name.

Chebyshev died in St Petersburg on 26 November 1894.

Mathematical contributions [edit]

Chebyshev is known for his work in the field of probability, statistics and number theory. Chebyshev's inequality says that if X is a random variable with standard deviation σ, the probability that the outcome of X is no less than $a\sigma$ away from its mean is no more than $1/a^2$:

$$\Pr(|X - \mathbf{E}(X)| \geq a\,\sigma) \leq \frac{1}{a^2}.$$

Chebyshev's inequality is used to prove the weak law of large numbers.

The Bertrand–Chebyshev theorem (1845|1850) states that for any $n > 1$, there exists a prime number p such that $n < p < 2n$. It is a consequence of Chebyshev inequalities for the number $\pi(n)$ of prime numbers less than n, which state that $\pi(n)$ is of the order of $n/\log(n)$. A more precise form is given by the celebrated prime number theorem: the *quotient* of the two expressions approaches 1 as n tends to infinity.

Legacy [edit]

Chebyshev is considered a founding father of Russian mathematics. Among his well-known students were the prolific mathematicians Dmitry Grave, Aleksandr Korkin, Aleksandr Lyapunov and Andrey Markov. According to the Mathematics Genealogy Project, Chebyshev has 7,483 mathematical

Фамилия - Чебышев или Чебышёв?

1. Inequalities involving means and standard deviations of ordered sequences. References: Hardy-Littlewood-Pólya, Mitrinovic.

Riesz means

- The counting function,

  $N(z) := \#(\lambda_k \le z)$
- Integrals of the counting function, known as *Riesz means*

$$R_\sigma(z) := \sum_j (z - \lambda_j)_+^\sigma$$

- Chandrasekharan and Minakshisundaram, 1952; Safarov, Laptev, Weidl, ...

Lemma 2.1 *Given any finite sequence* $\Sigma = \{x_1, \ldots, x_n\}$, *let* $\overline{x} := \frac{1}{n}\sum_n x_n$, $\overline{x^2} := \frac{1}{n}\sum_n x_n^2$, *and* $\sigma^2 := \overline{x^2} - \overline{x}^2$, *Then for each real number* z,

$$\begin{aligned}&\sum_{x_j \in J} z^2(\overline{x} - x_j) - z(\overline{x^2} - x_j^2) + x_j\overline{x^2} - x_j^2\overline{x} \\ &= \frac{1}{n}\sum_{x_j \in J}\sum_{x_k \in J^c}(z - x_j)(z - x_k)(x_k - x_j).\end{aligned} \tag{2.2}$$

As a consequence,

$$(z - \overline{x})\, R_2(z) \leq (z - \overline{x})^2\, R_1(z) + \sigma^2 R_1(z), \tag{2.3}$$

and

$$\frac{R_2(z)}{(z - \overline{x})^2 + \sigma^2}$$

is a nondecreasing function of z.

Proposition 2.1 *Suppose that for a nondecreasing sequence of numbers $\Sigma = \{x_1, x_2, \ldots\}$, and constants $C > 0$ and δ,*

$$R_2(z) \leq C \sum_j (z - x_j)_+ (x_j + \delta) \tag{2.1}$$

Then

$$\frac{R_2(z)}{(z+\delta)^{2+\frac{2}{C}}}$$

is a nondecreasing function of z.

If the sequence happens to be the spectrum of a self-adjoint matrix, then

$$\sum_{\lambda_j \in J} z^2(\mathrm{tr}(H) - n\lambda_j) - z(\mathrm{tr}(H^2) - n\lambda_j^2) + \lambda_j \mathrm{tr}(H^2) - \lambda_j^2 \mathrm{tr}(H)$$
$$= \sum_{\lambda_j \in J} \sum_{\lambda_k \in J^c} (z - \lambda_j)(z - \lambda_k)(\lambda_k - \lambda_j).$$

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$$tr(H) = \sum_v d_v = 2\mathcal{E}$$

$$tr(H^2) = \sum_v (d_v^2 + d_v) = 2(\mathcal{E} + \zeta)$$

$$tr(H^3) = \sum_u (d_v^3 + 3d_v^2) - 6T$$

How can a general identity give information about graphs?

$$\frac{R_2(z)}{z^2 - 2\mathcal{E}z + 2\zeta}$$

is a nondecreasing function of z.

This is sharp for complete graphs, and always has the limit 1, attained already for $z \geq \lambda_{n-1}$.

An analogue of Lieb-Thirring

- Consider the operator s Deg – A, which interpolates between -A and H as s goes from 0 to 1. Then (writing D for Deg)

$$-\frac{d}{ds}\sum(z-\lambda_j)_+^3 \leq 3(z^2\,\mathrm{tr}(D) - 2zs\,\mathrm{tr}(D^2) + s^2\mathrm{tr}(D^3) + \mathrm{tr}(D^2)).$$

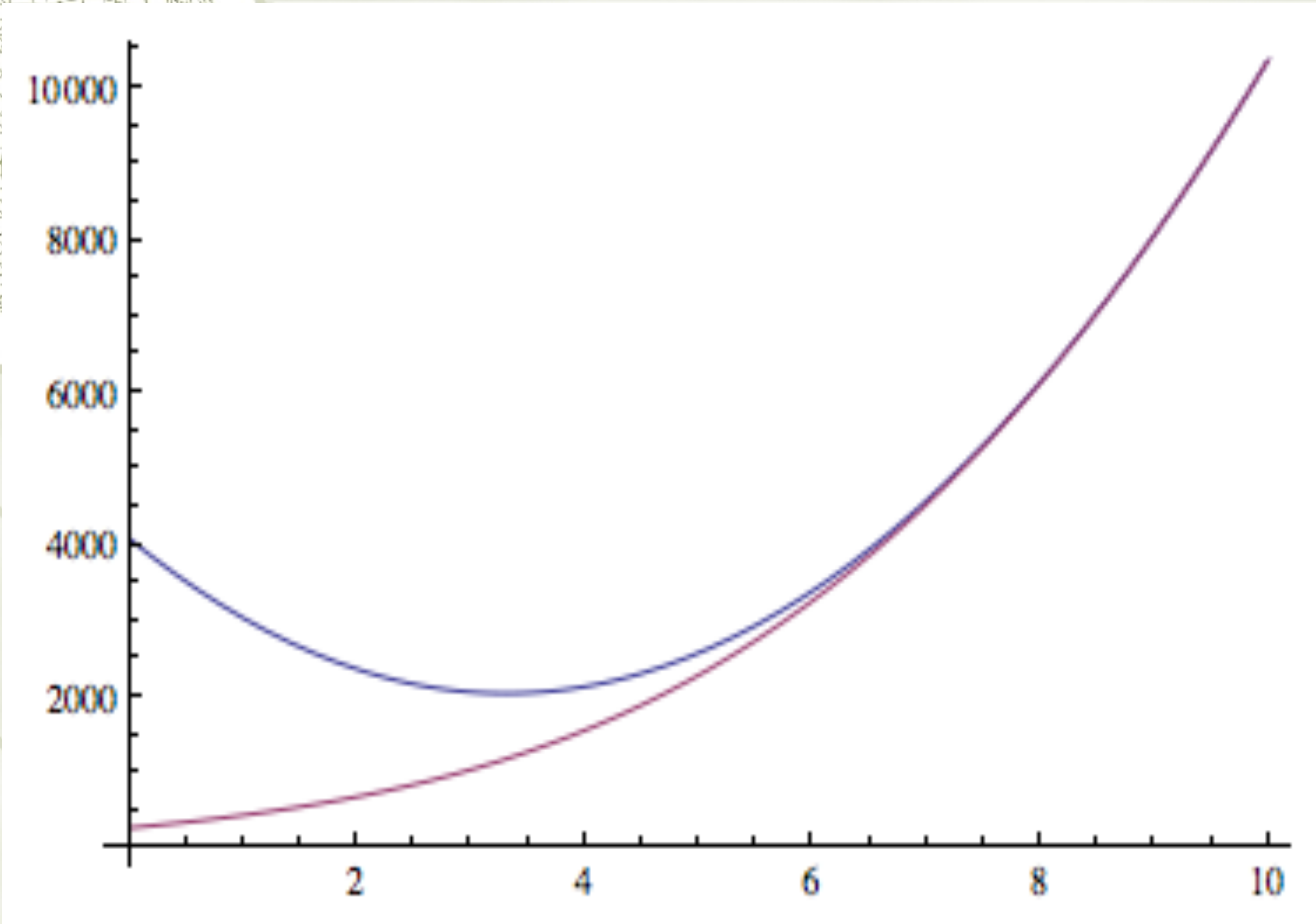
An analogue of Lieb-Thirring

- When integrated,

$$\mathrm{tr}((z+A)_+^3) - \mathrm{tr}((z-H)_+^3) \le 3z^2\,\mathrm{tr}(D) - 3z\,\mathrm{tr}(D^2) + \mathrm{tr}(D^3) + 3\mathrm{tr}(D^2)$$

i.e.,

$$\mathrm{tr}((z+A)_+^3) - \mathrm{tr}((z-H)_+^3) \le \mathrm{tr}((z+A)^3) - \mathrm{tr}((z-H)^3)$$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Algebraic methods

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Algebraic methods

1. Determinant calculations involving A date already to Kirchhoff.
2. In the context of Laplacians and Schrödinger operators, trace identities have been found useful for “universal inequalities” and semiclassical estimates (Harrell-Stubbe, Levitin-Parnovsky, Ashbaugh-Hermi, from 1990’s) Applying these methods to graphs is still a work in progress.

$$R_\sigma(z) := \sum_j (z - \lambda_j)_+^\sigma$$

1^{st} and 2^{nd} commutators

$$\frac{1}{2}\sum_{\lambda_j \in J} (z-\lambda_j)^2 \left\langle [G,[H,G]]\phi_j, \phi_j \right\rangle - \sum_{\lambda_j \in J} (z-\lambda_j) \| [H,G]\phi_j \|^2$$

$$=$$

$$\sum_{\lambda_j \in J} \sum_{\lambda_k \in J^c} (z-\lambda_j)(z-\lambda_k)(\lambda_k-\lambda_j) | \left\langle G\phi_j, \phi_k \right\rangle |^2$$

Harrell-Stubbe TAMS 1997

The only assumptions are that H and G are self-adjoint, and that the eigenfunctions are a complete orthonormal sequence. (If continuous spectrum, need a spectral integral on right.)

1^{st} and 2^{nd} commutators

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Harrell-Stubbe TAMS 1997

When does this side have a sign?

Take-away messages #1

1. There is an exact identity involving traces including [G, [H, G]] and [H,G]*[H,G].

2. For the lower part of the spectrum e hope for an inequality like:

$$\sum (z - \lambda_k)^{2} (\ldots) \quad \leq \quad \sum (z - \lambda_k) (\ldots)$$

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3. *******Once such an inequality is proved, the "usual correlaries," including universal gap and ratio bounds and Lieb-Thirring, follow*.

Recall the Dirichlet problem:

Trace identities imply differential inequalities

$$R_2(z) \le \frac{4}{d} \sum_k (z - \lambda_k) T_k$$

Harrell-Hermi JFA 08: Laplacian

$$\left(1 + \frac{4}{d}\right) R_2(z) - \frac{2z}{d} R_2'(z) \le 0.$$

Consequences – universal bound for k >j:

$$\frac{\overline{\lambda_k}}{\overline{\lambda_j}} \le \frac{4+d}{2+d} \left(\frac{k}{j}\right)^{2/d}$$

Statistics of spectra

$$\left(\left(1+\tfrac{2}{d}\right)\overline{\lambda_k}\right)^2-\left(1+\tfrac{4}{d}\right)\overline{\lambda_k^2}\geq 0.$$

A *reverse* Cauchy inequality:

The variance is dominated by the square of the mean.

For a given self-adjoint operator, the game is essentially:

1. Find a conjugate operator with Simple first and second commutators
2. Exploit differential inequalities and transforms to convert control over Riesz means into information about eigenvalues
3. *To get simple relations, you often need to perform an averaging.*

What are some good commutators?

1. Distance functions. These have the property that

$$[G,[H,G]] = A_{\widehat{G}},$$

where $\widehat{G}$ is always a spanning bipartite subgraph of G. As for the second commutator,

$$(-[G,H]^2)_{jk} = \pm\widehat{a}_{jk}.$$

What are some good commutators?

2. Projectors onto edges?

$$\mathrm{Ave}([G,[H,G]]) = \mathrm{Deg}A + A\mathrm{Deg}$$

$$\mathrm{Ave}(-[H,G]^2) = \mathrm{Deg}^2 + A\mathrm{Deg}A$$


THE END