UNIVERSAL INEQUALITIES FOR THE EIGENVALUES OF LAPLACE AND SCHRÖDINGER OPERATORS ON SUBMANIFOLDS

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Abstract. We establish inequalities for the eigenvalues of Schrödinger operators on compact submanifolds (possibly with nonempty boundary) of Euclidean spaces, of spheres, and of real, complex and quaternionic projective spaces, which are related to inequalities for the Laplacian on Euclidean domains due to Payne, Pólya, and Weinberger and to Yang, but which depend in an explicit way on the mean curvature. In later sections, we prove similar results for Schrödinger operators on homogeneous Riemannian spaces and, more generally, on any Riemannian manifold that admits an eigenmap into a sphere, as well as for the Kohn Laplacian on subdomains of the Heisenberg group.

Among the consequences of this analysis are an extension of Reilly's inequality, bounding any eigenvalue of the Laplacian in terms of the mean curvature, and spectral criteria for the immersibility of manifolds in homogeneous spaces.

1. introduction

Universal eigenvalue inequalities date from the work of Payne, Pólya, and Weinberger in the 1950's [27], who considered the Dirichlet problem for the Laplacian on a Euclidean domain. In this and similar cases, the term "universal" applies to expressions involving only the eigenvalues of a class of operators, without reference to the details of any specific operator in the class. Since that time the essentially purely algebraic arguments that lead to universal inequalities have been adapted in various ways for eigenvalues of differential operators on manifolds. (E.g., see [2, 7, 8, 15, 16, 21, 22, 26, 30, 33]. For a review of universal eigenvalue inequalities, we refer to [1, 3].) In particular, Ashbaugh and Benguria discussed universal inequalities for Laplacians on subdomains of hemispheres in [2], and Cheng and Yang have treated the case of Laplacians on minimal submanifolds of spheres [7, 8].

When either the geometry is more complicated or a potential energy is introduced, analogous inequalities must contain appropriate modifications. Our point of departure is a recent article [15], in which

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the eigenvalues of Schrödinger operators on hypersurfaces were studied and some trace identities and sharp inequalities were presented, containing the mean curvature explicitly. The goal of the present article is to further study the relation between the spectra of Laplacians or Schrödinger operators and the local differential geometry of submanifolds of arbitrary codimension. The approach is based on an algebraic technique which allows us to unify and extend many results in the literature (see [1, 3, 4, 15, 16, 21, 22, 26, 27, 32, 33] and Remarks 3.1, 4.1 5.1). There is an almost immediate extension of the results of [15] to the case of submanifolds of codimension greater than one, and because of the appearance of the mean curvature, we are able to generalize Reilly's inequality [28, 10, 11, 12] by bounding each eigenvalue of the Laplacian in terms of the mean curvature. In addition we derive the modifications necessary when the domain is contained in a submanifold of spheres, projective spaces, and certain other types of spaces. Finally, we are able to obtain some universal inequalities in the rather different context of the Kohn Laplacian on subdomains of the Heisenberg group.

Let M^n be a compact Riemannian manifold of dimension n, possibly with nonempty boundary ∂M, and let Δ be the Laplace–Beltrami operator on M. In the case where $\partial M \neq \emptyset$, Dirichlet boundary conditions apply (in the weak sense [9]). For any bounded real–valued potential q on M, the Schrödinger operator $H = -\Delta + q$ has compact resolvent (see [19, Theorem IV.3.17] and observe that a bounded q is relatively compact with respect to Δ). The spectrum of H consists of a nondecreasing, unbounded sequence of eigenvalues with finite multiplicities [5, 9]:

$$Spec(-\Delta + q) = \{\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_i \leq \cdots\}.$$

Notice that when $\partial M = \emptyset$ and $q = 0$, the zero eigenvalue is indexed by 1, that is $\lambda_1 = 0$.

To avoid technicalities, we suppose throughout that q is bounded, and that the mean curvature of the submanifolds under consideration is defined everywhere and bounded. Extensions to a wider class of potentials and geometries allowing singularities would not be difficult.

2. Submanifolds of $\mathbb{R}^m$

In this section M is either a closed Riemannian manifold or a bounded domain in a Riemannian manifold that can be immersed as a submanifold of dimension n of $\mathbb{R}^m$. The main theorem directly extends a result of [15], in which part (I) descends ultimately from a result of H.C. Yang for Euclidean domains [32, 17, 3, 4]:

Theorem 2.1. *Let $X : M \longrightarrow \mathbb{R}^m$ be an isometric immersion. We denote by h the mean curvature vector field of X (i.e the trace of its second fundamental form). For any bounded potential q on M, the* *spectrum of $H = -\Delta + q$ (with Dirichlet boundary conditions if $\partial M \neq \emptyset$) must satisfy, $\forall k \geq 1$,*

$$\text{(I)}\ \ n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\,(\lambda_i + \delta_i)$$

$$\text{(II)}\ \ \left(1 + \frac{2}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{2}{n}\frac{1}{k} \sum_{i=1}^{k} \delta_i - \sqrt{D_{nk}} \leq \lambda_{k+1} \leq \left(1 + \frac{2}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{2}{n}\frac{1}{k} \sum_{i=1}^{k} \delta_i + \sqrt{D_{nk}},$$

where u_i are the L^2–normalized eigenfunctions, $\delta_i := \int_M \left(\frac{|h|^2}{4} - q\right) u_i^2$, and

$$\text{(III)}\quad D_{nk} := \left(\left(1 + \frac{2}{n}\right) \frac{1}{k} \sum_{1}^{k} \lambda_i + \frac{2}{n}\frac{1}{k} \sum_{i=1}^{k} \delta_i\right)^2 - \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{1}^{k} \lambda_i^2 - \frac{4}{n}\frac{1}{k} \sum_{i=1}^{k} \lambda_i \delta_i \geq 0.$$

Theorem 2.1 can be simplified to eliminate all dependence on u_i with elementary estimates such as

$$\inf\left(\frac{|h|^2}{4} - q\right) \leq \delta_i \leq \sup\left(\frac{|h|^2}{4} - q\right). \tag{2.1}$$

Thus:

Corollary 2.1. *Under the circumstances of Theorem 2.1, $\forall k \geq 1$,*

$$\text{(I a)}\ \ n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\,(\lambda_i + \delta)$$

$$\text{(II a)}\ \ \lambda_{k+1} \leq \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{4\delta}{n}.$$

where $\delta := \sup\left(\frac{|h|^2}{4} - q\right)$.

Corollary 2.1, proved below, can be restated as a criterion for the immersibility of a manifold in $\mathbb{R}^m$:

Corollary 2.2. *Suppose that $\{\lambda_i\}$ are the eigenvalues of the Laplace–Beltrami operator on an abstract Riemannian manifold M of dimension n. If M is isometrically immersed in $\mathbb{R}^m$, then the mean curvature satisfies*

$$\|h\|_\infty^2 \geq n\lambda_{k+1} - \frac{(n+4)}{k} \sum_{i=1}^{k} \lambda_i \tag{2.2}$$

for each k.

Corollary 2.2 is representative of a large family of necessary conditions for immersibility in terms of the eigenvalues of Laplace–Beltrami and Schrödinger operators on M, which will not be presented in detail in this article. (See [15] for various sum rules on which such constraints can be based.)

Proof of Theorem 2.1. For a smooth function G on M, we will denote by G the multiplication operator naturally associated with G. To prove Theorem 2.1 we first need the following lemma involving the commutator of H and G.

Lemma 2.1. *For any smooth G and any positive integer k one has*

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\langle [H,G]u_i, Gu_i\rangle_{L^2} \leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\,\|[H,G]u_i\|_{L^2}^2 \quad (2.3)$$

This lemma dates from [17, Theorem 5], and in this form appears in [3, Theorem 2.1]. Variants can be found in [15, Corollary 4.3] and [23, Corollary 2.8].

Now, let $X_1, \dots X_m$ be the components of the immersion X. A straightforward calculation gives

$$[H, X_\alpha]u_i = [-\Delta, X_\alpha] = (-\Delta X_\alpha)u_i - 2\nabla X_\alpha \cdot \nabla u_i.$$

It follows by integrating by parts that

$$\langle [H, X_\alpha]u_i, X_\alpha u_i\rangle_{L^2} = \int_M |\nabla X_\alpha|^2 u_i^2.$$

Thus

$$\sum_\alpha \langle [H, X_\alpha]u_i, X_\alpha u_i\rangle_{L^2} = \sum_\alpha \int_M |\nabla X_\alpha|^2 u_i^2 = n\int_M u_i^2 = n.$$

On the other hand, we have

$$\|[H, X_\alpha]u_i\|_{L^2}^2 = \int_M \left((-\Delta X_\alpha)u_i - 2\nabla X_\alpha \cdot \nabla u_i)\right)^2.$$

Since X is an isometric immersion, it follows that $h = (\Delta X_1, \cdots, \Delta X_m)$, $\sum_\alpha (\nabla X_\alpha \cdot \nabla u_i)^2 = |\nabla u_i|^2$ and $\sum_\alpha (-\Delta X_\alpha)u_i \nabla X_\alpha \cdot \nabla u_i = h\cdot \nabla u_i^2 = 0$. Using all these facts, we get

$$\sum_\alpha \|[H, X_\alpha]u_i\|_{L^2}^2 = \int_M |h|^2 u_i^2 + 4\int_M |\nabla u_i|^2, \quad (2.4)$$

as in [15]. Then

$$\begin{aligned}\int_M |\nabla u_i|^2 &= \int_M u_i(-\Delta + q)u_i - \int_M q u_i^2 \\ &= \left(\lambda_i - \int_M q u_i^2\right).\end{aligned}$$

Using Lemma 2.1 we obtain

$$n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\int_M (|h|^2-4q)u_i^2 + 4\lambda_i\right)$$

which proves assertion (I) of Theorem 2.1.

From assertion (I) we get a quadratic inequality in the variable λ_{k+1}:

$$k\lambda_{k+1}^2 - \lambda_{k+1}\left(\left(2+\frac{4}{n}\right)\sum_{i=1}^{k}\lambda_i + \frac{4}{n}\sum_{i=1}^{k}\delta_i\right) + \left(1+\frac{4}{n}\right)\sum_{i=1}^{k}\lambda_i^2 + \frac{4}{n}\sum_{i=1}^{k}\lambda_i\delta_i \le 0 \tag{2.2}$$

The roots of this quadratic polynomial are the bounds in (II). The existence and reality of λ_{k+1} imply statement (III). □

Proof of Corollary 2.1. To derive (II a) from Theorem 2.1(II), it is simply necessary to replace δ_i by δ, and to note that the quantity D_{nk} is bounded above by

$$\left(\left(1+\frac{2}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{2\delta}{n}\right)^2 - \left(1+\frac{4}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i^2 - \frac{4\delta}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i,$$

which, since $\left(\sum_{i=1}^{k}\lambda_i\right) \le k\sum_{i=1}^{k}\lambda_i^2$, implies that

$$D_{nk} \le \left(\frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right)^2 + \left(\frac{2\delta}{n}\right)^2 + \frac{8\delta}{n^2}\frac{1}{k}\sum_{i=1}^{k}\lambda_i = \left(\frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{\delta}{2n}\right)^2,$$

with which the upper bound in (II) reduces to the right member of (II a). □

We observe next that Theorem 2.1 enables us to recover Reilly's inequality for λ_2 of the Laplace–Beltrami operator on closed submanifolds [11, 28]. Indeed, applying (I) with $k=1$, $\lambda_1=0$ and $u_1 = V^{-\frac{1}{2}}$, where V is the volume of M, we get

$$\lambda_2 \le \frac{4}{n}\,\delta_1 = \frac{1}{nV}\int_M |h|^2 \le \frac{1}{n}\,\|h\|_\infty^2\,.$$

Moreover, Theorem 2.1 allows extensions of Reilly's inequality to higher order eigenvalues. For example, the following corollary can be derived easily from Corollary 2.1(II a) by induction on k.

Corollary 2.3. *Under the circumstances of Theorem 2.1,* $\forall k \ge 2$,

$$\lambda_k \le \left(\frac{4}{n}+1\right)^{k-1}\lambda_1 + C_R(n,k)\,\|h\|_\infty^2\,,$$

where $C_R(n,k) = \frac{1}{4}\left((\frac{4}{n}+1)^{k-1}-1\right)$. *In particular, when* M *is closed and* $q=0$,

$$\lambda_k \le C_R(n,k)\,\|h\|_\infty^2\,. \tag{2.5}$$

The explicit value for the generalized Reilly constant $C_R(n,k)$ given in this corollary is likely far from optimal. We regard the sharp value of $C_R(n,k)$ as an interesting open problem. In the case of a minimally embedded submanifold of a sphere, Cheng and Yang claim a bound on λ_k ([7], eq. (1.23)) that scales like $k^{\frac{2}{n}}$ as in the Weyl law. We conjecture that $C_R(n,k)$ is sharply bounded by a constant times $k^{\frac{2}{n}}$ when $q=0$ and that when $q \neq 0, C_R(n,k)$ is correspondingly bounded by a semiclassical expression, as is the case for Schrödinger operators on flat spaces. (See, for instance, [31], section 3.5 and [24], part III).

In [15] it was argued that simplifications and optimal inequalities are obtained in some circumstances where M is a hypersurface and the potential q depends quadratically on curvature, a circumstance that arises naturally in the physics of thin structures ([13, 14] and references therein). In this spirit we close the section with some remarks for Schrödinger operators $H_g := -\Delta + g|h|^2$, for a real parameter g. As was already observed in [15], in view of (2.1), simplifications occur when $g=\frac{1}{4}$, rendering the quantities δ and δ_j given above zero.

Corollary 2.4. *Assume* M *is closed,* $|h|$ *is bounded, and* H *is of the form* H_g*, where* g *is an arbitrary real number. The inequalities* (I), (II)*, and* (III)*, in Theorem 2.1 are saturated (i.e., equalities) for all* k *such that* $\lambda_{k+1} \neq \lambda_k$*, if* M *is a sphere.*

Proof. We begin with the case of the Laplacian, $g=0$, for which the eigenvalues of the standard sphere $\mathbb{S}^n$ are known [25] to be $\{\ell(\ell+n-1)\}$, $\ell = 0,1,...,$ with multiplicities 1 for $\ell=0$; $n+1$ for $\ell=1$; and $\mu_{n,\ell} := \binom{n+\ell}{n} - \binom{n+\ell-2}{n} = \frac{(n(n+1)...(n+\ell-2))(n+2\ell-1)}{\ell!}$ thereafter. Thus $\lambda_1 = 0$, $\lambda_2 = \cdots = \lambda_{n+2} = n$, etc. , with gaps separating eigenvalues λ_k and λ_{k+1} when $k = \sum_{\ell=0}^m \mu_{n,\ell} = \frac{n+2m}{n}\binom{n+m-1}{m}$.

For the sphere, $\delta_j = \frac{n^2}{4}$, and an exact calculation shows, remarkably, that

$$n\sum_{i=1}^k (\lambda_{k+1}^{sphere} - \lambda_i^{sphere})^2 = \sum_{i=1}^k (\lambda_{k+1}^{sphere} - \lambda_i^{sphere})\left(4\lambda_i^{sphere} + n^2\right):$$

To see this, subtract $n\sum_{i=1}^k (\lambda_{k+1}^{sphere} - \lambda_i^{sphere})^2$ from the expression on the right and multiply the result by $(n-1)!$. After substitution and simplification, the expression reduces to

$$\sum_{\ell=1}^m \frac{((m-\ell+1)(n+m+\ell)(2\ell+n-1)(4\ell(\ell-1)-n^2(m-\ell)-n(m^2+m-\ell(\ell+3))(n+\ell-2)!}{\ell!},$$

which evaluates identically to 0. (Algebra was performed with the aid of Mathematica ™.)

This establishes equality in (I), and consequently (II) and (III) for this case. If $M = \mathbb{S}^n$, $|h|^2 = n^2$ is a constant, and if gn^2 is added to $-\Delta$, then each eigenvalue is shifted by the same amount and the left side of (I) is unchanged, as is the first factor in the sum on the right. As for the other factor, it becomes $\lambda_i + \delta_j = \ell(\ell+n-1) + gn^2 + \frac{n^2}{4} - gn^2$ and is likewise unchanged. It follows that the case of equality for H_g on the standard sphere persists for all g. □

3. Submanifolds of spheres and projective spaces

Theorem 2.1, together with the standard embeddings of sphere and projective spaces by means of the first eigenfunctions of their Laplacians, enables us to obtain results for immersed submanifolds of the latter. In what follows, $\mathbb{F}$ will denote the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers, or the field $\mathbb{Q}$ of quaternions. The m-dimensional projective space over $\mathbb{F}$ will be denoted by $\mathbb{F}P^m$; we endow it with its standard Riemannian metric so that the sectional curvature is either constant and equal to 1 ($\mathbb{F} = \mathbb{R}$) or pinched between 1 and 4 ($\mathbb{F} = \mathbb{C}$ or $\mathbb{Q}$). For convenience, we introduce the integers

$$d(\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F} = \begin{cases} 1 & \text{if } \mathbb{F} = \mathbb{R} \\ 2 & \text{if } \mathbb{F} = \mathbb{C} \\ 4 & \text{if } \mathbb{F} = \mathbb{Q}. \end{cases}$$

and

$$c(n) = \begin{cases} n^2, & \text{if } \overline{M} = \mathbb{S}^m \\ 2n(n + d(\mathbb{F})), & \text{if } \overline{M} = \mathbb{F}P^m. \end{cases} \tag{3.1}$$

Theorem 3.1. *Let $\overline{M}$ be $\mathbb{S}^m$ or $\mathbb{F}P^m$ and let $X : M \longrightarrow \overline{M}$ be an isometric immersion of mean curvature h. For any bounded potential q on M, the spectrum of $H = -\Delta_g + q$ (with Dirichlet boundary conditions if $\partial M \neq \emptyset$) must satisfy, $\forall k \in \mathbb{N}$, $k \geq 1$,*

(I) $n \sum_1^k (\lambda_{k+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left(\lambda_i + \bar{\delta}_i\right),$
where $\bar{\delta}_i := \frac{1}{4} \int_M (|h|^2 + c(n) - 4q) u_i^2,$

(II) $\lambda_{k+1} \leq \left(1 + \frac{2}{n}\right) \frac{1}{k} \sum_{i=1}^k \lambda_i + \frac{2}{n} \frac{1}{k} \sum_{i=1}^k \bar{\delta}_i + \sqrt{\bar{D}_{nk}}$

where

$$\bar{D}_{nk} := \left(\left(1 + \frac{2}{n}\right) \frac{1}{k} \sum_1^k \lambda_i + \frac{2}{n} \frac{1}{k} \sum_{i=1}^k \bar{\delta}_i\right)^2 - \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_1^k \lambda_i^2 - \frac{4}{n} \frac{1}{k} \sum_{i=1}^k \lambda_i \bar{\delta}_i \geq 0,$$

A lower bound is also possible along the lines of Theorem 2.1. As in the previous section, the following simplifications follow easily:

Corollary 3.1. *With the notation of Theorem 3.1 one has,* $\forall k \geq 1$,

$$\lambda_{k+1} \leq \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{4}{n}\, \bar{\delta},$$

where $\bar{\delta} := \frac{1}{4}\sup\left(|h|^2 + c(n) - 4q\right)$.

Moreover, as in the discussion for Corollary 2.4, when M is a submanifold of a sphere or projective space, a simplification occurs in Theorem 3.1 and Corollary 3.1 when $q(x) = \frac{1}{4}(|h|^2 + c(n))$, in that the curvature and potential do not appear explicitly at all.

Remark 3.1. *Theorems 2.1 and 3.1 and corollaries 2.1 and 3.1 unify and extend many results in the literature (see* [1, 3, 4, 15, 16, 21, 22, 26, 27, 32, 33] *and the references therein). In particular, the recent results of Cheng and Yang* [7] *and* [8] *concerning the eigenvalues of the Laplacian on*

- *a domain or a minimal submanifold of* $\mathbb{S}^m$
- *a domain or a complex hypersurface of* $\mathbb{C}P^m$

respectively, appear as particular cases of Theorem 3.1. Recall that a complex submanifold of $\mathbb{C}P^m$ *is automatically minimal (that is,* $h = 0$*).*

Proof of Theorem 3.1. We will treat separately the cases $\overline{M} = \mathbb{S}^m$ and $\overline{M} = \mathbb{F}P^m$.

Immersed submanifolds of a sphere:

Let $\bar{M} = \mathbb{S}^m$ and denote by i the standard embedding of $\mathbb{S}^m$ into $\mathbb{R}^{m+1}$. We have

$$|h(i \circ X)|^2 = |h(X)|^2 + n^2.$$

Applying Theorem 2.1 to the isometric immersion $i \circ X : (M, g) \to \mathbb{R}^{m+1}$, we obtain the result.

Immersed submanifolds of a projective space:

First, we need to recall some facts about the first standard embeddings of projective spaces into Euclidean spaces (see for instance [6, 29, 30] for details). Let $\mathcal{M}_{m+1}(\mathbb{F})$ be the space of $(m+1) \times (m+1)$ matrices over $\mathbb{F}$ and set $\mathcal{H}_{m+1}(\mathbb{F}) = \{A \in \mathcal{M}_{m+1}(\mathbb{F}) \mid A^* :={}^t \bar{A} = A\}$ the subspace of Hermitian matrices. We endow $\mathcal{M}_{m+1}(\mathbb{F})$ with the inner product given by

$$\langle A, B\rangle = \frac{1}{2} tr(A\, B^*).$$

For A, $B \in \mathcal{H}_{m+1}(\mathbb{F})$, one simply has $\langle A, B\rangle = \frac{1}{2} tr(A\, B)$.

The first standard embedding $\varphi : \mathbb{F}P^m \to \mathcal{H}_{m+1}(\mathbb{F})$ is defined as the one induced via the canonical fibration $\mathbb{S}^{(m+1)d-1} \to \mathbb{F}P^m$ $(d := d(\mathbb{F}))$, from the natural immersion $\psi : \mathbb{S}^{(m+1)d-1} \subset \mathbb{F}^{m+1} \longrightarrow \mathcal{H}_{m+1}(\mathbb{F})$ given by

$$\psi(z) = \begin{pmatrix} |z_0|^2 & z_0\bar{z}_1 & \cdots & z_0\bar{z}_m \\ z_1\bar{z}_0 & |z_1|^2 & \cdots & z_1\bar{z}_m \\ \cdots & \cdots & \cdots & \cdots \\ z_m\bar{z}_0 & z_m\bar{z}_1 & \cdots & |z_m|^2 \end{pmatrix}.$$

The embedding φ is isometric and the components of $\varphi - \frac{1}{m+1}I$ are eigenfunctions associated with the first eigenvalue of the Laplacian of $\mathbb{F}P^m$ (see, for instance, [30] for details). Hence, $\varphi(\mathbb{F}P^m)$ is a minimal submanifold of the hypersphere $\mathbb{S}(\sqrt{m/2(m+1)})$ of $\mathcal{H}_{m+1}(\mathbb{F})$ centered at $\frac{1}{m+1}I$.

Lemma 3.1. *Let $X : M \to \mathbb{F}P^m$ be an isometric immersion and let h and h' be the mean curvature vector fields of the immersions X and $\varphi \circ X$ respectively. Then we have*

$$|h'|^2 = |h|^2 + \frac{4n(n+2)}{3} + \frac{2}{3}\sum_{i\neq j} K(e_i, e_j)$$

where K is the sectional curvature of $\mathbb{F}P^m$ and $(e_i)_{i\leq n}$ is a local orthonormal frame tangent to $X(M)$.

We refer to [6, 29], or [30] for a proof of this lemma.

Now, from the expression of the sectional curvature of $\mathbb{F}P^m, \forall i \neq j$ we get

- $K(e_i, e_j) = 1$ if $\mathbb{F} = \mathbb{R}$.
- $K(e_i, e_j) = 1 + 3\,(e_i \cdot Je_j)^2$, where J is the almost complex structure of $\mathbb{C}P^m$, if $\mathbb{F} = \mathbb{C}$.
- $K(e_i, e_j) = 1 + \sum_{r=1}^{3} 3\,(e_i \cdot J_r e_j)^2$, where (J_1, J_2, J_3) is the almost quaternionic structure of $\mathbb{Q}P^m$, if $\mathbb{F} = \mathbb{Q}$.

Thus in the case of $\mathbb{R}P^m$, we obtain $|h'|^2 = |h|^2 + 2n(n+1)$. For $\mathbb{C}P^m$, we get

$$\begin{aligned} |h'|^2 &= |h|^2 + 2n(n+1) + 2\sum_{i,j}(e_i \cdot Je_j)^2 \\ &= |h|^2 + 2n(n+1) + 2\left\|J^T\right\|^2 \leq |h|^2 + 2n(n+2), \qquad (3.2) \end{aligned}$$

where J^T is the tangential part of the almost complex structure J of $\mathbb{C}P^m$. Indeed, we clearly have $\left\|J^T\right\|^2 \leq n$, where the equality holds if and only if $X(M)$ is a complex submanifold of $\mathbb{C}P^m$. For the case of

$\mathbb{Q}P^m$, we obtain similarly

$$\begin{aligned} |h'|^2 &= |h|^2 + 2n(n+1) + 2\sum_{i,j}\sum_{r=1}^{3}(e_i \cdot J_r e_j)^2 \\ &= |h|^2 + 2n(n+1) + 2\sum_{r=1}^{3}||J_r^T||^2 \leq |h|^2 + 2n(n+4), \quad (3.3) \end{aligned}$$

where $(J_r^T)_{1\leq r\leq 3}$ are the tangential components of the almost quaternionic structure of $\mathbb{Q}P^m$. The equality in (3.3) holds if and only if $n \equiv 0 \pmod 4$ and $X(M)$ is an invariant submanifold of $\mathbb{Q}P^m$.

To finish the proof of Theorem 3.1, it suffices to apply Theorem 2.1 to the isometric immersion $\varphi \circ X$ of M in the Euclidean space $\mathcal{H}_{m+1}(\mathbb{F})$ using the inequalities (3.2) and (3.3). □

Remark 3.2. *It is worth noticing that in some special geometrical situations, the constant $c(n)$ in the inequalities of Theorem 3.1 and corollary 3.1 can be replaced by a sharper one. For instance, when $\bar{M} = \mathbb{C}P^m$ and*

- *M is odd–dimensional, then one can replace $c(n)$ by $c'(n) = 2n(n+2-\frac{1}{n})$,*
- *$X(M)$ is totally real (that is $J^T = 0$), then $c(n)$ can be replaced by $c'(n) = 2n(n+1)$.*

Indeed, under each one of these assumptions, the estimate of $\|J^T\|^2$ by n (see the inequality (3.2) above) can be improved by elementary calculations.

4. Manifolds admitting spherical eigenmaps

Let (M, g) be a compact Riemannian manifold. A map

$$\varphi = (\varphi_1 \dots, \varphi_{m+1}) : (M, g) \longrightarrow \mathbb{S}^m$$

is termed an *eigenmap* if its components $\varphi_1 \dots, \varphi_{m+1}$ are all eigenfunctions associated with the same eigenvalue λ of the Laplacian of (M, g). Equivalently, an eigenmap is a harmonic map with constant energy density $(\sum_\alpha |\nabla\varphi_\alpha|^2 = \lambda)$ from (M, g) into a sphere. In particular, any minimal and homothetic immersion of (M, g) into a sphere is an eigenmap. Moreover, a compact homogeneous Riemannian manifold without boundary admits eigenmaps for all the positive eigenvalues of its Laplacian (see for instance [22]).

We still denote by $\{u_i\}$ a complete L^2-orthonormal basis of eigenfunctions of H associated to $\{\lambda_i\}$.

Theorem 4.1. *Let λ be an eigenvalue of the Laplacian of (M, g) and assume that (M, g) admits an eigenmap associated with the eigenvalue λ. Then, for any bounded potential q on M, the spectrum of $H = -\Delta_g + q$ (with Dirichlet boundary conditions if $\partial M \neq \emptyset$) must satisfy, $\forall k \in \mathbb{N}$, $k \geq 1$,*

(I) $\displaystyle\sum_1^k(\lambda_{k+1}-\lambda_i)^2 \le \sum_{i=1}^k(\lambda_{k+1}-\lambda_i)\left(\lambda+4\left(\lambda_i-\int_M qu_i^2\right)\right).$

(II) $\displaystyle\lambda_{k+1} \le (1+\frac{2}{n})\frac{1}{k}\sum_{i=1}^k\lambda_i+\frac{(\lambda-4\inf q)}{2n}+\frac{\sqrt{\hat D_{nk}}}{2nk}.$

where

$$\hat D_{nk} = (2(n+2)\sum_1^k\lambda_i+k(\lambda-\inf q))^2-4nk((n+4)\sum_1^k\lambda_i^2+(\lambda-\inf q)A)$$

Corollary 4.1. *Let (M,g) be a compact homogeneous Riemannian manifold without boundary. The inequalities of Theorem 4.1 hold, λ being here the first positive eigenvalue of the Laplacian of (M,g).*

Remark 4.1. *Theorem 4.1 and Corollary 4.1 are to be compared to results of* [7, 16, 22].

Proof of Theorem 4.1. Let $\varphi=(\varphi_1\dots,\varphi_{m+1}):(M,g)\to\mathbb{S}^m$ be a λ-eigenmap. As in the proof of Theorem 2.1, we use Lemma 2.1 with $G=\varphi_\alpha,\ \alpha=1,2,\dots,m+1$, to obtain

$$\sum_\alpha\sum_{i=1}^k(\lambda_k-\lambda_i)^2\langle[H,\varphi_\alpha]u_i,\varphi_\alpha u_i\rangle_{L^2}\le\sum_\alpha\sum_{i=1}^k(\lambda_k-\lambda_i)\left\|[H,\varphi_\alpha]u_i\right\|_{L^2}^2.$$

A direct computation gives

$$[H,\varphi_\alpha]u_i=\lambda\varphi_\alpha u_i-2\nabla\varphi_\alpha\cdot\nabla u_i$$

and

$$\langle[H,\varphi_\alpha]u_i,\varphi_\alpha u_i\rangle_{L^2}=\lambda\int_M\varphi_\alpha^2u_i^2-\frac{1}{2}\int_M\nabla\varphi_\alpha^2\cdot\nabla u_i^2.$$

Summing up, we obtain

$$\sum_\alpha\langle[H,\varphi_\alpha]u_i,\varphi_\alpha u_i\rangle_{L^2}=\lambda,$$

since $\sum_\alpha\varphi_\alpha^2$ is constant. Since $\sum_\alpha|\nabla\varphi_\alpha|^2=\lambda$ and $\int_M|\nabla u_i|^2=\lambda_i-\int_M qu_i^2$, the same kind of calculation yields

$$\begin{aligned}\sum_\alpha\left\|[H,\varphi_\alpha]u_i\right\|_{L^2}^2=\lambda^2 &+ 4\sum_\alpha\int_M(\nabla\varphi_\alpha\cdot\nabla u_i)^2\\ &\le \lambda^2+4\int_M\sum_\alpha|\nabla\varphi_\alpha|^2|\nabla u_i|^2\\ &= \lambda(\lambda+4(\lambda_i-\int_M qu_i^2).\end{aligned}$$

In conclusion, we have

$$\lambda\sum_1^k(\lambda_{k+1}-\lambda_i)^2\le\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)(\lambda^2+4\lambda(\lambda_i-\int_M qu_i^2)),$$

which gives the first assertion of Theorem 4.1. We derive the second assertion as in the proof of Theorem 2.1.

□

5. Applications to the Kohn Laplacian on the Heisenberg group.

Let us recall that the $2n+1$-dimensional Heisenberg group $\mathbb{H}^n$ is the space $\mathbb{R}^{2n+1}$ equipped with the non-commutative group law

$$(x,y,t)(x',y',t') = (x+x', y+y', t+t'+\frac{1}{2}(\langle x',y\rangle_{\mathbb{R}^n} - \langle x,y'\rangle_{\mathbb{R}^n}),$$

where $x,x',y,y' \in \mathbb{R}^n$, t and t' $\in \mathbb{R}$. Its Lie algebra $\mathcal{H}^n$ has as a basis the vector fields

$$\{T = \frac{\partial}{\partial t},\ X_i = \frac{\partial}{\partial x_i} + \frac{y_i}{2}\frac{\partial}{\partial t},\ Y_i = \frac{\partial}{\partial y_i} - \frac{x_i}{2}\frac{\partial}{\partial t}\ ;\ i \le n\}.$$

We observe that the only non–trivial commutators are $[X_i, Y_j] = -T\delta_{ij},\ i,j = 1,\cdots,n$. Let $\Delta_{\mathbb{H}^n}$ denote the real Kohn Laplacian (or the sublaplacian associated with the basis $\{X_1,\cdots,X_n,Y_1,\cdots,Y_n\}$):

$$\Delta_{\mathbb{H}^n} = \sum_{i=1}^n X_i^2 + Y_i^2 = \Delta_{xy}^{\mathbb{R}^{2n}} + \frac{1}{4}(|x|^2+|y|^2)\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t}\sum_{i=1}^n \left(y_i\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial y_i}\right).$$

We shall be concerned with the following eigenvalue problem :

$$-\Delta_{\mathbb{H}^n} u = \lambda u \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega. \tag{5.1}$$

where Ω is a bounded domain of the Heisenberg group $\mathbb{H}^n$ with smooth boundary. It is known that the Dirichlet problem (5.1) has a discrete spectrum. The Kohn Laplacian dates from [20], and the problem (5.1) has been studied, e.g., in [18, 26]. We denote its eigenvalues by

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \cdots \to +\infty,$$

and orthonormalize its eigenfunctions $u_1,\ u_2,\ \cdots \in S_0^{1,2}(\Omega)$ so that, $\forall i,j \ge 1$,

$$\langle u_i, u_j\rangle_{L^2} = \int_\Omega u_i u_j dx\, dy\, dt = \delta_{ij}.$$

Here, $S^{1,2}(\Omega)$ denotes the Hilbert space of the functions $u \in L^2(\Omega)$ such that $X_i(u)$, $Y_i(u) \in L^2(\Omega)$, and $S_0^{1,2}$ denotes the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the Sobolev norm

$$||u||^2_{S^{1,2}} = \int_\Omega (|\nabla_{\mathbb{H}^n} u|^2 + |u|^2) dx\, dy\, dt,$$

with $\nabla_{\mathbb{H}^n} u = (X_1(u),\cdots,X_n(u),Y_1(u),\cdots,Y_n(u))$.

We shall prove a result similar to Theorem 2.1 for the problem (5.1):

Theorem 5.1. *For any* $k \ge 1$

(I) $n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq 2 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i$

(II) $(\frac{n+1}{nk}) \sum_{i=1}^{k} \lambda_i - \sqrt{D} \leq \lambda_{k+1} \leq (\frac{n+1}{nk}) \sum_{i=1}^{k} \lambda_i + \sqrt{\tilde{D}_{nk}}$

where $\tilde{D}_{nk} = \left(\left(1 + \frac{1}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i\right)^2 - \left(1 + \frac{2}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i^2 \geq 0.$

Remark 5.1. *Using the Cauchy–Schwarz inequality* $(\sum_{i=1}^{k} \lambda_i)^2 \leq k \sum_{i=1}^{k} \lambda_i^2$, *we deduce from Theorem 5.1 (II) that*

$$\lambda_{k+1} \leq \left(\frac{1}{k} + \frac{2}{nk}\right) \left(\sum_{i=1}^{k} \lambda_i\right)$$

which improves a result of Niu and Zhang [26].

Proof. The key observation here is that Lemma 2.1 remains valid for $H = L = -\Delta_{\mathbb{H}^n}$ and $G = x_\alpha$ or $G = y_\alpha$. Thus we have

$$\sum_{i=1}^{k} \sum_{\alpha=1}^{n} (\lambda_{k+1} - \lambda_i)^2 (\langle [L, x_\alpha] u_i, x_\alpha u_i \rangle_{L^2} + \langle [L, y_\alpha] u_i, y_\alpha u_i \rangle_{L^2}) \leq \sum_{i=1}^{k} \sum_{\alpha=1}^{n} (\lambda_{k+1} - \lambda_i)(\| [L, x_\alpha] u_i \|_{L^2}^2 + \| [L, y_\alpha] u_i \|_{L^2}^2) \tag{5.2}$$

with

$$[L, x_\alpha]\, u_i = -2X_\alpha(u_i) \text{ and } [L, y_\alpha]\, u_i = -2Y_\alpha(u_i).$$

Thus,

$$\sum_{\alpha=1}^{n} \| [L, x_\alpha] u_i \|_{L^2}^2 + \| [L, y_\alpha] u_i \|_{L^2}^2 = 4 \int_\Omega |\nabla_{\mathbb{H}^n} u_i|^2 = 4\lambda_i.$$

Now, using the skew-symmetry of X_α (resp. Y_α), we have

$$\int_\Omega X_\alpha(u_i)\, x_\alpha u_i = -\int_\Omega u_i X_\alpha(x_\alpha u_i) = -\int_\Omega u_i^2 - \int_\Omega X_\alpha(u_i) x_\alpha u_i$$

and the same identity holds with y_α and Y_α. Therefore,

$$-2 \int_\Omega X_\alpha(u_i) x_\alpha u_i = -2 \int_\Omega Y_\alpha(u_i) y_\alpha u_i = \int_\Omega u_i^2 = 1.$$

We put these identities in (5.2) and obtain the first assertion of Theorem 5.1. The second assertion follows as in the proof of Theorem 2.1. □

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