

On the fundamental eigenvalue ratio of the p -Laplacian

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Abstract

It is shown that the fundamental eigenvalue ratio $\frac{\lambda_2}{\lambda_1}$ of the p -Laplacian is bounded by a quantity depending only on the dimension N and p .

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I. Introduction

The linear Laplacian on a domain or a manifold can be realized as a self-adjoint operator, and the theory of its spectrum is a well developed subject [4,5]. For the p -Laplacian on a domain Ω , with vanishing Dirichlet boundary conditions, it has been known since the work of Anane and Tsouli [1] that a sequence of real eigenvalues can be defined by a variational procedure analogous to the min-max principle for the linear case $p = 2$, but many rather basic questions about the spectrum remain to be addressed. For background on the p -Laplacian, which arrives from the first variation of the functional

$$\frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} |u|^p}, \quad (1)$$

we refer to [8, 9, 16]. In this article Ω will be a smooth, bounded Euclidean domain, and u will vary in the Sobolev space $W_0^{1,p}(\Omega)$, corresponding to Dirichlet conditions on the boundary.

Several useful estimates are available in the linear case for the lowest two eigenvalues, especially the lowest, or fundamental, eigenvalue, and some of these have been extended to the nonlinear cases. For instance, see [16, 17, 18]. In this article we seek information on the ratio of the first two eigenvalues, which to our knowledge has not been much studied except when $p = 2$, for which case it is known, for example, that the ratio is bounded universally above by the ratio attained when Ω is a ball [2].

We are guided by some earlier analysis of the case $p = 2$ in [11, 12, 13] as well as [2, 3], but the unavailability of the spectral theorem eliminates some essential parts of the analysis. Therefore we attempt to rely instead on certain integral inequalities as in [10].

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Let us recall some properties of the Dirichlet p -Laplacian:

1. The p -Laplacian is defined for $u \in W^{1,p}(\Omega)$ by $\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} |\nabla u|)$. (Here we always equip the p -Laplacian with zero Dirichlet boundary conditions.)
2. It is then natural to define an eigenvalue λ_j as a value of λ for which the eigenvalue problem

$$-\Delta_p u := \lambda |u|^{p-2} u \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega \quad (2)$$

has a nontrivial solution.

3. It is not, however, known whether every such quantity is a variational eigenvalue as for the case $p = 2$. Of course it is also possible to consider the ‘‘variational eigenvalues’’ but it is not known in general whether these numbers coincide. Nevertheless, it has long been known that the first eigenvalue λ_1 , which is isolated and simple, is the infimum, for $u \in W_0^{1,p}(\Omega)$, of the ‘‘Rayleigh quotient’’ defined by (1).
4. The infimum is achieved for a multiple of a function φ_1 that may be chosen positive on Ω . Henceforth we impose the normalization $\int_\Omega \varphi_1^p = 1$. The pair (λ_1, φ_1) is referred to as the ‘‘principal eigenpair.’’
5. In 1996 Anane and Tsouli [1] gave a characterization of the variational eigenvalues of the Dirichlet p -Laplacian For any positive integer j , let

$$\lambda_j := \inf_{\mathcal{C} \in \mathcal{C}_j} \max_{u \in \mathcal{C}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}, \quad (3)$$

with $\mathcal{C}_j = \{\mathcal{C} \in W_0^{1,p}(\Omega)/\mathcal{C} = -\mathcal{C}; \gamma(\mathcal{C}) \geq j\}$, where γ denotes the Krasnosel’skii genus [14]. (By definition, the Krasnosel’skii genus of a radially symmetric subset \mathcal{A} of a Banach space \mathcal{B} is the smallest integer j for which there exists a nontrivial continuous odd mapping $\mathcal{A} \rightarrow \mathbb{R}^j - \{\mathbf{0}\}$.)

6. Finally, it was proved by Anane and Tsouli that λ_2 defined by (3) is effectively the second eigenvalue of the Dirichlet p -Laplacian defined on Ω , in the sense that the eigenvalue problem (2) has no other eigenvalue between λ_1 and λ_2 . Hence the ratio λ_2/λ_1 is well defined and amenable to variational analysis.

Upper bounds on the fundamental eigenvalue ratio

When $p = 2$, bounds on λ_2/λ_1 can be derived from variational estimates of the gap $\lambda_2 - \lambda_1$. When $p \leq 2$, the same will be true here in certain situations, but for $p > 2$ we are led by Lemma 3.1 in [10] to consider instead the difference $\lambda_2 - \hat{k}\lambda_1$ with a suitable constant $\hat{k} \geq 1$. In [10] constants (depending on p and the dimension N) were defined as:

$$m^p := \max_{0 \leq x \leq 1} ((p-x)x^{p-1} + (1-x)^p), \quad (4)$$

and

$$\begin{array}{lll} p \geq 2, & N = 1: & \hat{m} = m = p - 1 & \hat{k} = k = p^{2-p}(p-1)^{p-1} \\ p \geq 2, & N \geq 2: & \hat{m} = 2^{\frac{(p-2)}{2p}}(p-1) & \hat{k} = 2^{\frac{(p-2)}{2}}p^{2-p}(p-1)^{p-1} \\ 1 < p \leq 2, & N = 1: & \hat{m} = m & \hat{k} = k = 1 \\ 1 < p \leq 2, & N \geq 2: & \hat{m} = 2^{\frac{(2-p)}{2p}}m & \hat{k} = 1 \end{array} \quad (5)$$

Our main result is

Theorem 1. *Let Ω be a smooth, N -dimensional bounded Euclidean domain, and denote the two lowest Dirichlet eigenvalues for the p -Laplacian by λ_1, λ_2 .*

If $p \leq 2$ and $N > p$, then

$$\Gamma := \lambda_2 - \lambda_1 \leq \hat{m}^p N \left(\frac{p}{N-p} \right)^p \lambda_1, \quad (6)$$

or, equivalently,

$$\frac{\lambda_2}{\lambda_1} \leq \left[1 + \hat{m}^p \left(\frac{p}{N-p} \right)^p N \right]. \quad (7)$$

For $p \geq 2$,

$$\Gamma := \lambda_2 - \hat{k} \lambda_1 \leq \hat{m}^p N^{-\frac{p}{2}} p^p \lambda_1, \quad (8)$$

or, equivalently,

$$\frac{\lambda_2}{\lambda_1} \leq \left[\hat{k} + \hat{m}^p N^{-\frac{p}{2}} p^p \right]. \quad (9)$$

We prepare the proof with two estimates. The first is a standard (and known) uncertainty-principle inequality:

Lemma 2. *Given a bounded domain $\Omega \in \mathbb{R}^N$, with $N > p$, for any $u \in W_0^{1,p}(\Omega)$,*

$$\int_{\Omega} \left| \frac{u}{\|\mathbf{x}\|_2} \right|^p \leq \left(\frac{p}{N-p} \right)^p \int_{\Omega} |\nabla u|_2^p,$$

where $\mathbf{x} = (x_1, \dots, x_N)$ are the Cartesian coordinates and where $\|\mathbf{x}\|_p := (x_1^p + \dots + x_N^p)^{1/p}$. In particular,

$$\int_{\Omega} \left| \frac{\varphi_1}{\|\mathbf{x}\|_2} \right|^p \leq \left(\frac{p}{N-p} \right)^p \lambda_1, \quad (10)$$

This follows easily from an inequality of Boggio, as shown in [10].

Lemma 3. *Given a bounded domain $\Omega \in \mathbb{R}^N$, if $p \geq 2$,*

$$\frac{1}{\int_{\Omega} \|\mathbf{x}\|_2^p \varphi_1^p} \leq \left(\frac{p}{N} \right)^p \lambda_1.$$

Proof.

$$\begin{aligned} 1 &= \int_{\Omega} \varphi_1^p = \frac{1}{N} \int_{\Omega} \varphi_1^p \nabla \cdot \mathbf{x} = -\frac{p}{N} \int_{\Omega} \mathbf{x} \cdot \varphi_1^{p-1} \nabla \varphi_1 \\ &\leq \frac{p}{N} \left[\int_{\Omega} |\nabla \varphi_1|^p \right]^{1/p} \left[\int_{\Omega} \|\mathbf{x}\|_2^{p'} \varphi_1^p \right]^{1/p'}, \end{aligned}$$

with $p' = \frac{p}{p-1}$ as usual, by Hölder's inequality. Therefore

$$1 \leq \left(\frac{p}{N} \right)^p \lambda_1 \left[\int_{\Omega} \|\mathbf{x}\|_2^{p'} \varphi_1^p \right]^{p-1}.$$

If $p = 2$, this establishes the claim. Otherwise, $p > 2$, that is, $p' < p$, and by Hölder's inequality,

$$\begin{aligned}
1 &\leq \left(\frac{p}{N}\right)^p \lambda_1 \left[\int_{\Omega} (\|\mathbf{x}\|_2 \varphi_1)^{p'} \varphi_1^{\frac{p(p-2)}{p-1}} \right]^{p-1} \\
&\leq \left(\frac{p}{N}\right)^p \lambda_1 \left[\int_{\Omega} \|\mathbf{x}\|_2^p \varphi_1^p \right] \left[\int_{\Omega} \varphi_1^p \right]^{p-2} \\
&= \left(\frac{p}{N}\right)^p \lambda_1 \left[\int_{\Omega} \|\mathbf{x}\|_2^p \varphi_1^p \right]. \quad \square
\end{aligned}$$

Proof of Theorem 1. For a unified treatment, we write $\Gamma := \lambda_2 - \hat{k}\lambda_1$, noting that for $p \leq 2$, $\hat{k} = 1$. Let δ be a given real constant and set $\omega := \{x = (x_1, \dots, x_N) \in \Omega, \text{ with } x_k < \delta\}$. Assume that δ is chosen so that $\text{meas}(\omega) > 0$ and $\text{meas}(\Omega \setminus \omega) > 0$. In other words $\delta_{\min} < \delta < \delta_{\max}$. Now choose $g \in \mathcal{C}_0^1(\Omega)$ such that $g(x)_{x_k=\delta} = 0$. We define

$$C := \{\varphi_1 \cdot G_{\alpha,\beta} : \alpha \in \mathbb{R}, \beta \in \mathbb{R}; |\alpha|^p + |\beta|^p = 1\},$$

where

$$G_{\alpha,\beta} := g(x)(\alpha\chi_{\omega} + \beta\chi_{\Omega \setminus \omega}). \quad (12)$$

It is easy to see that the set C has the following properties;

- (1) $C = -C$ (change α to $-\alpha$ and β to $-\beta$).
- (2) $\gamma(C) = 2$.

From the variational characterization (4), for any k , any δ , and any function $g \in \mathcal{C}_0^1(\Omega)$ satisfying $g(x)|_{x_k=\delta} = 0$, it follows that

$$\lambda_2 \leq \max_{|\alpha|^p + |\beta|^p = 1} R(\alpha, \beta), \quad (13)$$

where

$$R(\alpha, \beta) := \frac{N(\alpha, \beta)}{D(\alpha, \beta)}$$

with

$$N(\alpha, \beta) := \int_{\Omega} |\nabla(\varphi_1 G_{\alpha,\beta})|^p$$

and

$$D(\alpha, \beta) := \int_{\Omega} |\varphi_1 G_{\alpha,\beta}|^p.$$

From Lemma 3.1 in [10], it follows that

$$\begin{aligned}
N(\alpha, \beta) &\leq \hat{m}^p \int_{\Omega} |\nabla(G_{\alpha,\beta})\varphi_1|^p + \hat{k} \int_{\Omega} \varphi_1 |G_{\alpha,\beta}|^p (-\Delta_p \varphi_1) \\
&= \hat{m}^p \int_{\Omega} |\nabla(G_{\alpha,\beta})\varphi_1|^p + \hat{k}\lambda_1 \int_{\Omega} |G_{\alpha,\beta}|^p \varphi_1^p.
\end{aligned}$$

Therefore:

$$\Gamma = \lambda_2 - \hat{k}\lambda_1 \leq \hat{m}^p \max_{|\alpha|^p + |\beta|^p = 1} \frac{|\alpha|^p \int_{\omega} |\varphi_1 \nabla g|^p + |\beta|^p \int_{\Omega \setminus \omega} |\varphi_1 \nabla g|^p}{|\alpha|^p \int_{\omega} |\varphi_1 G_{\alpha, \beta}|^p + |\beta|^p \int_{\Omega \setminus \omega} |\varphi_1 G_{\alpha, \beta}|^p},$$

supposing that $G_{\alpha, \beta} = 0$ on $\{x_j = \delta\}$. The maximization in this expression is elementary: writing t for $|\alpha|^p$, the problem is to maximize an expression of the form $\frac{at+b(1-t)}{ct+d(1-t)}$ for $0 \leq t \leq 1$. Unless this expression is constant and equal to $\frac{b}{d} = \frac{a}{c} = \frac{a+b}{c+d}$, its derivative is always nonzero. If not constant, it is therefore maximized when $t = 0$ as $\frac{b}{d}$ or else when $t = 1$ as $\frac{a}{c}$. We conclude that

$$\Gamma \leq \hat{m}^p \max \left\{ \frac{\int_{\omega} \varphi_1^p}{\int_{\omega} \varphi_1^p |x_j - \delta|^p}, \frac{\int_{\Omega \setminus \omega} \varphi_1^p}{\int_{\Omega \setminus \omega} \varphi_1^p |x_j - \delta|^p} \right\}. \quad (14)$$

Observe that each of the integrals in (14) depends continuously on δ , and that as δ approaches the minimal value of x_j in Ω , $\frac{\int_{\omega} \varphi_1^p}{\int_{\omega} \varphi_1^p |x_j - \delta|^p} \rightarrow +\infty$, while $\frac{\int_{\Omega \setminus \omega} \varphi_1^p}{\int_{\Omega \setminus \omega} \varphi_1^p |x_j - \delta|^p}$ remains bounded. The converse is the case as δ approaches the maximal value of x_j in Ω . By continuity, there is a value of δ for which

$$\frac{\int_{\omega} \varphi_1^p}{\int_{\omega} \varphi_1^p |x_j - \delta|^p} = \frac{\int_{\Omega \setminus \omega} \varphi_1^p}{\int_{\Omega \setminus \omega} \varphi_1^p |x_j - \delta|^p} = \frac{\int_{\Omega} \varphi_1^p}{\int_{\Omega} \varphi_1^p |x_j - \delta|^p}.$$

Hence

$$\int_{\Omega} \varphi_1^p |x_j - \delta|^p \leq \frac{\hat{m}^p}{\Gamma}.$$

Let us henceforth choose the origin of the coordinate system so that $\delta = 0$, and then sum on j , obtaining

$$\int \varphi_1^p \|\mathbf{x}\|_p^p \leq \frac{\hat{m}^p}{\Gamma} \cdot N.$$

Therefore

$$\Gamma \leq \frac{\hat{m}^p N}{\int_{\Omega} \varphi_1^p \|\mathbf{x}\|_p^p}. \quad (15)$$

Suppose now that $N > p$ and $p \leq 2$. According to the Cauchy–Schwarz inequality,

$$1 = \left(\int_{\Omega} \varphi_1^p \right)^2 = \left(\int_{\Omega} \varphi_1^{p/2} \|\mathbf{x}\|_p^{p/2} \varphi_1^{p/2} \|\mathbf{x}\|_p^{-p/2} \right)^2 \leq \int_{\Omega} \varphi_1^p \|\mathbf{x}\|_p^p \int_{\Omega} \varphi_1^p \|\mathbf{x}\|_p^{-p}. \quad (16)$$

We recall that since the dimension N is finite and since $p \leq 2$, we have $\frac{1}{\|\mathbf{x}\|_p} \leq \frac{1}{\|\mathbf{x}\|_2}$.

Hence, from (16) and (10) we derive:

$$1 \leq \int_{\Omega} \varphi_1^p \|\mathbf{x}\|_p^p \cdot \int_{\Omega} \frac{\varphi_1^p}{\|\mathbf{x}\|_2^p} \leq \left(\frac{p}{N-p} \right)^p \lambda_1 \int_{\Omega} \varphi_1^p \|\mathbf{x}\|_p^p. \quad (17)$$

Combining (17) with (15), we derive (6) and (7) for $p \leq 2$.

In case $p \geq 2$, a tighter bound can be derived. Since for $p \geq 2$, $\frac{1}{\|\mathbf{x}\|_p} \leq N^{\frac{p-2}{2p}} \frac{1}{\|\mathbf{x}\|_2}$, we deduce from Inequality (15) that

$$\Gamma \leq \frac{\hat{m}^p N^{\frac{p}{2}}}{\int_{\Omega} \varphi_1^p \|\mathbf{x}\|_2^p}. \quad (18)$$

Combining (18) with Lemma 3, we derive Inequalities (8) and (9). \square

It is reasonable to ask how sharp is this bound. Unfortunately, other than in the one-dimensional (or radial) case, eigenvalues of the p -Laplacian are only known numerically, and then essentially only the principal eigenvalue [15]. A comparison is possible in one-dimension, where the eigenvalues are known explicitly [6, 7] and $\frac{\lambda_2}{\lambda_1} = 2^p$. Since the Hardy constant in one dimension is $\frac{p}{p-1}$, this compares to our bound of $p^{2-p}(p-1)^{p-1} + ((p-1)p)^p$ for $p \geq 2$. For $p = 2$ it is reasonably sharp (5 rather than 4), but for higher values of p it is less so.

As a final remark, we observe that it has been shown recently in [14] that for the first two eigenvalues Λ_1, Λ_2 of the Lindqvist ∞ -eigenvalue problem, $\Lambda_k = \lim_{p \rightarrow \infty} \lambda_k(p)^{1/p}$. As a direct consequence of Theorem 1, we therefore have:

Corollary 4.

$$\lim_{p \rightarrow \infty} \frac{1}{p} \left(\frac{\lambda_2(p)}{\lambda_1(p)} \right)^{1/p} \leq \frac{\hat{m}}{\sqrt{N}}.$$

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